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Galois G-covering of quotients of linear categories

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ABSTRACT

In this paper, we introduce the notion of G-liftable ideals, which extends the liftable ideas defined by Assem and Le Meur. We characterize the G-liftable ideals and construct the Galois G-coverings of quotients of categories associated to the G-liftable ideals. In particular, we study the behavior of G-liftable admissible ideals under Galois G-coverings. Furthermore, we show that the ideals generated by finite dimensional projective modules over a locally bounded linear categories are admissible G-liftable ideals. As an application, we provide a reduction technique for dealing with the existence of Serre functors in the stable categories of Gorenstein projective objects.

1. Introduction

Covering theory originated from topology theory, which is widely used in algebraic topology [15]. Covering technique was introduced into representation theory of algebras and developed by Riedtmann [22], Bongartz-Gabriel [7], Gabriel [14], Dowbor-Lenzing-Skowroński [10], Martínez-Villa, de la Peña [20], Cibils-Eduardo [8], et al. The classical Galois covering technique has been playing an important role in the representation theory of finite dimensional algebras. Let $F : \mathcal{A} \to \mathcal{B}$ be a classical Galois covering of locally bounded linear categories. The most important result in classical Galois covering theory is the Gabriel's theorem which shows that $\mathcal{A}$ is locally representation-finite if and only if $\mathcal{B}$ is so. This theory reduces problems of $\mathcal{B}$ whose structure is more complicated to that of $\mathcal{A}$, which is easier to treat and better understood.

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Let k be an algebraically closed field. Let $\mathcal{A}$ be a k-linear category. Denote by $\mathcal{A}_0$ the objects classes of $\mathcal{A}$. Recall that a k-linear category $\mathcal{A}$ is said to be *locally bounded* if it satisfies the following conditions:

(1) $\mathcal{A}$ is *basic* (i.e., $x \neq y \Rightarrow x \ncong y$);
(2) $\mathcal{A}$ is *semiperfect* (i.e., $\mathcal{A}(x,x)$ is a local algebra, $\forall\, x \in \mathcal{A}_0$);
(3) For each $x \in \mathcal{A}_0$, $\sum_{y\in\mathcal{A}_0} \dim_k \mathcal{A}(x,y) < \infty$ and $\sum_{y\in\mathcal{A}_0} \dim_k \mathcal{A}(y,x) < \infty$.

The classical Galois covering technique requires the stringent conditions on categories, such as, categories are locally bounded and group action is free. It makes very inconvenient to apply the covering technique to usual additive categories such as the bounded derived categories of the module category or even the module category. To overcome these difficulties, Asashiba [2] introduced the notion of a G-precovering and called a dense G-precovering i.e. "G-covering", which remove all stringent conditions on categories. He showed that a G-covering is a universal "G-invariant" functor.

Bautista and Liu [6] defined the notion of a Galois G-covering for general linear categories, which is a special kind of G-coverings. They showed that given a Galois G-covering F, a morphism f is radical if and only if $F(f)$ is radical. Moreover, Darpö and Iyama [11] showed that for a Galois G-covering $F : \mathcal{A} \to \mathcal{B}$ between Krull-Schmidt categories, F induces an isomorphism

$$\oplus_{g\in G}\mathrm{rad}_{\mathcal{A}}(x, gy) \cong \mathrm{rad}_{\mathcal{B}}(F(x), F(y)).$$

From this point, the radical of $\mathcal{A}$ can be regarded as the "étale" of the radical of $\mathcal{B}$. Based on these results, they showed that a Galois G-covering between Krull-Schmidt categories preserves almost split sequences. Recently, Asashiba, Hafezi and Vahed [1] provided G-precoverings of bounded derived categories, singularity categories and Gorenstein defect categories. Then they obtained a Gorenstein version of Gabriel's theorem. Hafezi, Mahdavi [16] showed that there is a G-precoverings between the stable categories and extended naturally the push-down functor to the G-precovering between the corresponding (mono)morphism categories. Using these results, they gave a (mono)morphism category version of Gabriel's theorem.

Recently, Assem and Le Meur [3] introduced the notion of F-liftable ideals with respect to the Galois covering functor F. They gave some characterizations of F-liftable ideals and construct the Galois coverings of quotients of categories associated to the F-liftable ideals.

Inspired by this notion, we aim to introduce the notion of G-liftable ideals with respect to a G-precovering functor F (here, "G" is a double entendre, which means both "generalization" and the group G), which generalizes the F-liftable ideals and can be applied to more situations. Under our settings, the concept contains as much as possible some well-known ideals. For example, for a Galois G-covering functor $F : \mathcal{A} \to \mathcal{B}$, the radical of $\mathcal{B}$ is a G-liftable ideal. Moreover, for a Galois covering functor $\pi : \mathcal{A} \to \mathcal{B}$, it is well-known that the push-down functor $\pi_\bullet : \mathsf{mod}\mathcal{A} \to \mathsf{mod}\mathcal{B}$ is a G-precovering. We will show that $\mathsf{prj}\mathcal{B}$ is a G-liftable ideal with respect to the push-down functor $\pi_\bullet$.

Note that stable categories of module categories and Gorenstein projective objects are both quotients of categories associated to the ideals generated by projective objects. Therefore, we aim to show that there is an induced Galois G-covering functor between quotients of categories associated to G-liftable ideals. Then, we can reprove the corresponding results in [16] and [1]. Remarkable that admissible ideal in the sense of [19] are closely relative to almost split morphisms and Serre functors. Thus, we also consider the behavior of G-liftable admissible ideals under Galois G-coverings. Using the induced Galois G-covering functor by G-liftable admissible ideals, we can study the existence of Serre functors for triangulated categories.

The paper is organized as follows: In Section 2, we collect some basic definitions and properties of Galois G-coverings. In Section 3, we introduce G-liftable ideals and give some characterizations. In Section 4, we discuss the behavior of G-liftable admissible ideals under Galois G-coverings. In Section 5, we give a class of G-liftable admissible ideals for module categories. Meanwhile, we apply the above results to get the reduction technique of the existence of Serre functors in the stable categories of Gorenstein projective objects.

2. Preliminaries

Throughout this paper, all categories are skeletally small, and morphisms are composed from the right to the left. Let R be a commutative artin ring. An *R-linear category* (or simply, linear category) is a category in which the morphism sets are R-modules such that the composition of morphisms is R-bilinear. All functors between R-linear categories are assumed to be R-linear. An R-linear category is called *additive* if it has finite direct sums.

A linear category is called *Hom-finite* if the morphism modules are of finite R-length. Moreover, a *Krull-Schmidt* category is an additive category in which every non-zero object is a finite direct sum of objects with a local endomorphism algebra. An additive category has *split idempotents* if every idempotent endomorphism ϕ of an object x splits, that is, there exists a factorization $x \xrightarrow{u} y \xrightarrow{v} x$ of ϕ with $uv = id_y$ and $vu = \phi$.

The following results are well-known.

Lemma 2.1 *([17, Corollary 4.4], [9, Theorem A.1]). Let $\mathcal{A}$ be an additive category. The following hold.*

(1) *$\mathcal{A}$ is a Krull-Schmidt category if and only if it has split idempotents and the endomorphism ring of every object is semi-perfect.*
(2) *Suppose that $\mathcal{A}$ is a Krull-Schmidt category. Let $u : x \to y$ and $v : y \to x$ be morphisms in $\mathcal{A}$. Suppose $x \cong y$.*
    (a) *If u is a retraction, then u is an isomorphism.*
    (b) *If v is a section, then v is an isomorphism.*

Proof. It is enough to show (2). Since u is a retraction, there exists a morphism $v : y \to x$ such that $uv = id_y$. Let $e = vu$. Then both e and $id_x - e$ are idempotent endomorphisms of x. Note that $\mathcal{A}$ has split idempotents. There exists $y' \in \mathcal{A}_0$, $u' : x \to y'$ and $v' : y' \to x$ such that $v'u' = id_x - e$ and $u'v' = id_{y'}$. Thus, $x \cong y \oplus y'$. Since $\mathcal{A}$ is a Krull-Schmidt category and $x \cong y$, we know that $y' = 0$. Therefore, $vu = e = id_x$. It means that u is an isomorphism. In this case, so does v. □

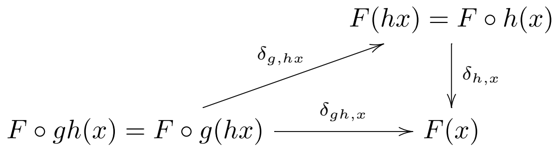
Let $\mathcal{A}$ be a linear category equipped with an action of a group G, that is, there exists a group homomorphism $\rho : G \to \mathrm{Aut}(\mathcal{A})$, where $\mathrm{Aut}(\mathcal{A})$ is the group of automorphisms of $\mathcal{A}$. Set $gx := \rho(g)(x)$, $gf := \rho(g)(f)$ for any $g \in G$, $x, y \in \mathcal{A}_0$ and $f \in \mathcal{A}(x, y)$. By abuse of notation, we identify g with $\rho(g)$.

Definition 2.2 *([6]).* Let $\mathcal{A}$ be a linear category with G a group acting on $\mathcal{A}$. The G-action on $\mathcal{A}$ is called *admissible* if it satisfies the following conditions

(1) G-action is *free*, that is $gx \ncong x$, for any indecomposable object x of $\mathcal{A}$ and any non-identity $g \in G$.
(2) G-action is *locally bounded*, that is for any indecomposable objects x, y of $\mathcal{A}$, $\mathcal{A}(x, gy) = 0$ for all but finitely many $g \in G$.

Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between linear categories. Recall that for $g \in G$, a functorial (iso)morphism $\delta_g : F \circ g \to F$ consists of (iso)morphisms $\delta_{g,x} : F \circ g(x) \to F(x)$ for any $x \in \mathcal{A}_0$, which are natural in x.

Definition 2.3 *([2]).* Let $\mathcal{A}$, $\mathcal{B}$ be linear categories with G a group acting on $\mathcal{A}$. A functor $F : \mathcal{A} \to \mathcal{B}$ is called *G-stable* provided there exist functorial isomorphisms $\delta_g : F \circ g \to F$, with $g \in G$, such that the following diagram commutative



that is $\delta_{h,x}\delta_{g,hx} = \delta_{gh,x}$ for any $g, h \in G$ and $x \in \mathcal{A}_0$. In this case, we call $\delta = (\delta_g)_{g\in G}$ a *G-stabilizer* for F.

Remark 2.4 *([6]).*

(1) By definition, $\delta_{g,x}^{-1} = \delta_{g^{-1},gx}$ for $g \in G$ and $x \in \mathcal{A}_0$; $\delta_e = id_F$, where e is the identity of G.
(2) F is said to be *G-invariant* if the G-stabilizer δ satisfies $\delta_g = id_F$ for any $g \in G$.

Definition 2.5 *([2]).* Let $\mathcal{A}$, $\mathcal{B}$ be linear categories with G a group acting on $\mathcal{A}$. A functor $F : \mathcal{A} \to \mathcal{B}$ is called a *G-precovering* provided that F has a G-stabilizer such that, for any X, $Y \in \mathcal{A}_0$, the following two maps are isomorphisms:

$$F_{x,y} : \bigoplus_{g\in G} \mathcal{A}(x, gy) \longrightarrow \mathcal{B}(F(x), F(y)) : \ (u_g)_{g\in G} \mapsto \sum_{g\in G} \delta_{g,y} \circ F(u_g)$$

$$F^{x,y} : \bigoplus_{g\in G} \mathcal{A}(gx, y) \longrightarrow \mathcal{B}(F(x), F(y)) : \ (v_g)_{g\in G} \mapsto \sum_{g\in G} F(v_g) \circ \delta_{g,x}^{-1}.$$

Definition 2.6 *([2]).* Let $\mathcal{A}$, $\mathcal{B}$ be linear categories with G a group acting on $\mathcal{A}$. A G-precovering $F : \mathcal{A} \to \mathcal{B}$ is called a G-covering provided that F is *dense*, in sense that for any $x' \in \mathcal{B}_0$, there exists an $x \in \mathcal{A}_0$ such that x' is isomorphic to $F(x)$ in $\mathcal{B}$.

Definition 2.7 *([14]).* Let $\mathcal{A}$, $\mathcal{B}$ be locally bounded categories with G a group acting admissible on $\mathcal{A}$. A functor $F : \mathcal{A} \to \mathcal{B}$ is called a *Galois covering* if F satisfies the following conditions.

(1) F is a G-covering.
(2) F is G-invariant.
(3) G acts transitively of the fiber $F^{-1}(x)$ for any $x \in \mathcal{B}_0$.

Definition 2.8 *([4]).* Let $\mathcal{A}$, $\mathcal{B}$ be linear categories with G a group acting admissible on $\mathcal{A}$. A G-covering functor F is called *Galois G-covering* if F satisfies the following conditions.

(G1) If $x \in \mathcal{A}_0$ is indecomposable, then $F(x)$ is indecomposable.
(G2) If $x, y \in \mathcal{A}_0$ are indecomposable with $F(x) = F(y)$, then there exists some $g \in G$ such that $y = gx$.

Remark 2.9 *([6]).* If $\mathcal{A}$, $\mathcal{B}$ are locally bounded linear categories over an algebraically closed field, then a Galois covering $F : \mathcal{A} \to \mathcal{B}$ is simply a G-invariant Galois G-covering.

Lemma 2.10 *([6]). Let $\mathcal{A}$, $\mathcal{B}$ be linear categories with G a group acting on $\mathcal{A}$ and let $F : \mathcal{A} \to \mathcal{B}$ be a G-precovering with a G-stabilizer δ.*

(1) *For any x, $y \in \mathcal{A}_0$, we have the following decompositions*

$$\mathcal{B}(F(x), F(y)) = \bigoplus_{g\in G} \delta_{g,y} F(\mathcal{A}(x, gy)) = \bigoplus_{h\in G} F(\mathcal{A}(hx, y))\delta_{h,x}^{-1}.$$

(2) *The functor F is faithful.*

Remark 2.11. By the direct sum decompositions in Lemma 2.10 (1), for any $u \in \mathcal{B}(F(x), F(y))$, we can write $u = \sum_{g\in G} \delta_{g,y} F(u_g) = \sum_{h\in G} F(v_h)\delta_{h,x}^{-1}$, where $u_g \in \mathcal{A}(x, gy)$ and $v_h \in \mathcal{A}(hx, y)$ such that $u_g = 0$ and $v_h = 0$ for all but finitely many g and $h \in G$, respectively.

Lemma 2.12 *([6]). Let $\mathcal{A}$, $\mathcal{B}$ be linear categories with G a group acting on $\mathcal{A}$ and let $F : \mathcal{A} \to \mathcal{B}$ be a G-precovering. Consider a morphism $u : X \to Y$ in $\mathcal{A}$.*

(1) *If $v : X \to Z$ or $v : Z \to Y$ is a morphism in $\mathcal{A}$, then v factorizes through u if and only if $F(v)$ factorizes through $F(u)$.*
(2) *The morphism u is a section, retraction, or isomorphism if and only if $F(u)$ is a section, retraction, or isomorphism, respectively.*

Now, assume that $\mathcal{A}$ is an additive category. The *radical* $\mathrm{rad}_{\mathcal{A}}(-,-)$ of $\mathcal{A}$ is the (two-sided) ideal of $\mathcal{A}$ defined by

$$\mathrm{rad}_{\mathcal{A}}(X,Y) := \{\ f \in \mathrm{Hom}_{\mathcal{A}}(X,Y) \mid id_X - gf \text{ is invertible for each } g : Y \to X\}$$

for any two objects $X, Y \in \mathcal{A}$. A morphism $f : X \to Y$ is said to be radical if $f \in \mathrm{rad}_{\mathcal{A}}(X,Y)$. Furthermore, $\mathrm{rad}_{\mathcal{A}}(X,X) \subseteq \mathsf{End}_{\mathcal{A}}(X)$ coincides with the Jocobson radical $J(\mathsf{End}_{\mathcal{A}}(X))$ of the ring $\mathsf{End}_{\mathcal{A}}(X)$.

Given $m \geq 1$, we recall that the m-th power $\mathrm{rad}^m_{\mathcal{A}}(-,-)$ of $\mathrm{rad}_{\mathcal{A}}(-,-)$ by taking for $\mathrm{rad}^m_{\mathcal{A}}(X,Y)$ the subspace of $\mathrm{rad}_{\mathcal{A}}(X,Y)$ consisting of all finite sums of morphisms of the form

$$X = X_0 \xrightarrow{h_1} X_1 \xrightarrow{h_2} X_2 \to \cdots \to X_{m-1} \xrightarrow{h_m} X_m = Y$$

where $h_i \in \mathrm{rad}_{\mathcal{A}}(X_{i-1}, X_i)$. See [5,17,18] for more details.

Proposition 2.13 *([6, Lemma 3.1 and 3.2]). Let $\mathcal{A}$, $\mathcal{B}$ be two additive categories with G a group acting admissibly on $\mathcal{A}$ and let $F : \mathcal{A} \to \mathcal{B}$ be a Galois G-covering. Then the following statements hold.*

(1) *Let $u \in \mathcal{A}(X,Y)$. Then u is radical if and only if $F(u)$ is radical.*
(2) *Let $u \in \mathcal{B}(F(X), F(Y))$. In this case, one may write $u = \sum_{g\in G} \delta_{g,Y} F(u_g)$, where $u_g \in \mathcal{A}(X, gY)$ such that $u_g = 0$ for all but finitely many $g \in G$. If m is a positive integer, then $u \in \mathrm{rad}^m_{\mathcal{B}}(F(X), F(Y))$ if and only if $u_g \in \mathrm{rad}^m_{\mathcal{A}}(X, gY)$ for all $g \in G$.*

Let $\mathcal{A}$ be an additive category. Recall that $\mathcal{I}$ is said to be an ideal on $\mathcal{A}$ if $\mathcal{I}(x,y)$ is an R-submodule of $\mathcal{A}(x,y)$ and for any $f \in \mathcal{I}(x,y)$, $g \in \mathcal{A}(z,x)$ and $h \in \mathcal{A}(y,s)$, $fg \in \mathcal{I}(z,y)$ and $hf \in \mathcal{I}(x,s)$. Then the quotient category of $\mathcal{A}$, denoted by $\mathcal{A}/\mathcal{I}$, has the same objects as $\mathcal{A}$, and for any two objects $x, y \in \mathcal{A}_0$, $\mathcal{A}/\mathcal{I}(x,y) := \mathcal{A}(x,y)/\mathcal{I}(x,y)$ is the quotient module of $\mathcal{A}(x,y)$. For f a morphism in $\mathcal{A}$, we denote by $\overline{f}$ its residue class in the quotient category. It is obviously that the quotient category $\mathcal{A}/\mathcal{I}$ is additive. It is well-known that if $\mathcal{A}$ is Hom-finite Krull-Schmidt, then $\mathcal{A}/\mathcal{I}$ is also Hom-finite Krull-Schmidt and $\mathrm{ind}(\mathcal{A}/\mathcal{I}) = \mathrm{ind}(\mathcal{A})\backslash\{x \in \mathrm{ind}(\mathcal{A}) \mid id_x \in \mathcal{I}(x,x)\}$.

Example 2.14. Let $\mathcal{A}$ be an additive category. $\mathcal{I}$ a full subcategory of $\mathcal{A}$ which is closed under taking direct sums and direct summands, (i.e., for any two objects $x, y \in \mathcal{A}_0$, $x \oplus y \in \mathcal{I}$ if and only if x, $y \in \mathcal{I}$) and $\mathcal{I}(x,y)$ is an R-submodule of $\mathcal{A}(x,y)$ consisting of morphisms factoring through some object in $\mathcal{I}$. Then, in this case, $\mathcal{I}$ is an ideal on $\mathcal{A}$, see [13, Lemma 4.3].

The following example is due to [12, Example 4.26].

Example 2.15 *(Ideal generated by a class of morphisms).* Let

$$\mathcal{R} = \{\psi_\lambda : x_\lambda \to y_\lambda \mid \lambda \in \Lambda\}$$

be a nonempty class of morphisms in an additive category $\mathcal{A}$. The ideal generated by $\mathcal{R}$ is the ideal $\mathcal{I}$ of $\mathcal{A}$ defined as follows. For any $x,\ y \in \mathcal{A}_0$, we define the subgroup

$$\mathcal{I}(x, y) = \{u \in \mathcal{A}(x, y) \mid u = \Sigma_{i=1}^n \upsilon_i \psi_{\lambda_i} \omega_i,\ \text{for some } \psi_{\lambda_i} \in \mathcal{R},\ \upsilon_i \in \mathcal{A}(y_{\lambda_i}, y) \text{ and } \omega_i \in \mathcal{A}(x, x_{\lambda_i})\}.$$

The ideal generated by $\mathcal{R}$ is the smallest ideal $\mathcal{I}$ of $\mathcal{A}$ such that $\psi_\lambda \in \mathcal{I}(x_\lambda, y_\lambda)$ for every $\lambda \in \Lambda$.

3. The G-covering of quotient categories

Definition 3.1. Let $\widehat{\mathcal{A}}$, $\mathcal{A}$ be linear categories with G a group acting on $\widehat{\mathcal{A}}$. Assume that $F : \widehat{\mathcal{A}} \to \mathcal{A}$ is a G-precovering. The ideal $\mathcal{I}$ on $\mathcal{A}$ is said to be *G-liftable* with respective to F, (or simply, *G-liftable*) if for any two objects $x, y \in \widehat{\mathcal{A}}_0$ and $u \in \mathcal{I}(F(x), F(y))$, then $F(u_g) \in \mathcal{I}(F(x), F(gy))$ for any $g \in G$, where $(u_g)_{g\in G} \in \bigoplus_{g\in G} \widehat{\mathcal{A}}(x, gy)$ such that $u = \sum_{g\in G} \delta_{g,y} \circ F(u_g)$.

Remark 3.2. If $\widehat{\mathcal{A}}$, $\mathcal{A}$ are two locally bounded k-category with G a group acting on $\widehat{\mathcal{A}}$ and $F : \widehat{\mathcal{A}} \to \mathcal{A}$ is a Galois covering, then the F-liftable ideal in sense of [3] is simply G-liftable.

Example 3.3. Let $\mathcal{A}$ and $\mathcal{B}$ be Krull-Schmidt categories with G a group acting admissibly on $\mathcal{A}$. Let $F : \mathcal{A} \to \mathcal{B}$ be a Galois G-covering with a G-stabilizer δ. Then, from [6, Lemma 3.1 and Lemma 3.2], $\operatorname{rad}^n(\mathcal{B})$ is G-liftable, for any integer $n \geq 1$.

Proposition 3.4. *Let $\widehat{\mathcal{A}}$, $\mathcal{A}$ be linear categories with G a group acting on $\widehat{\mathcal{A}}$, $\mathcal{I}$ a ideal on $\mathcal{A}$. Assume that $F : \widehat{\mathcal{A}} \to \mathcal{A}$ is a G-covering. The ideal $\mathcal{I}$ is G-liftable if and only if there exists a class of morphisms $\mathcal{R} \subseteq \bigcup_{x,y\in\widehat{\mathcal{A}}_0} F(\widehat{\mathcal{A}}(x, y))$ such that $\mathcal{I}$ is generated by $\mathcal{R}$.*

Proof. To prove the necessity, we set the following class of morphisms

$$\mathcal{R} = \bigcup_{g\in G, x,y\in\widehat{\mathcal{A}}_0} \Big\{F(u_g) : F(x) \to F \circ g(y) \ \Big|\ u \in \mathcal{I}(F(x), F(y))\Big\},$$

where, as before, $(u_g)_{g\in G} \in \bigoplus_{g\in G} \widehat{\mathcal{A}}(x, gy)$, such that $u = \Sigma_{g\in G} \delta_{g,y} F(u_g)$. Since $\mathcal{I}$ is Galois G-liftable, each morphism as the form of $F(u_g) : F(x) \to F \circ g(y)$ in $\mathcal{R}$ lies in $\mathcal{I}$. Then, $\mathcal{R} \subseteq \mathcal{I}$. For any two objects $x,\ y \in \mathcal{A}_0$, as F is dense, there exist two objects $x',\ y' \in \widehat{\mathcal{A}}_0$, such that $x \cong F(x')$ and $y \cong F(y')$. Hence, for any morphism $h \in \mathcal{I}(x, y)$, $u = ahb \in \mathcal{I}(F(x'), F(y'))$, where $a : y \to F(y')$ and $b : F(x') \to x$ are isomorphisms. Since $F : \widehat{\mathcal{A}} \to \mathcal{A}$ is a G-precovering, by Lemma 2.10 (1), we may write $u = \sum_{i=1}^n \delta_{g_i,y'} F(u_{g_i})$, where $g_1, \cdots, g_n$ are distinct and $u_{g_i} \in \widehat{\mathcal{A}}(x', g_i y')$, for $1 \leq i \leq n$. Since $\mathcal{I}$ is Galois G-liftable, $F(u_{g_i}) \in \mathcal{I}(F(x'), F g_i(y')) \in \mathcal{R}$. Then $h = \sum_{i=1}^n a^{-1} \delta_{g_i,y'} F(u_{g_i}) b^{-1}$. It means that $\mathcal{I}$ is generated by $\mathcal{R}$.

For the sufficiency, we assume that $\mathcal{I}$ is generated by $\mathcal{R} \subseteq \bigcup_{x,y\in\widehat{\mathcal{A}}_0} F(\widehat{\mathcal{A}}(x, y))$. Fix a pair $x,\ y \in \widehat{\mathcal{A}}_0$, and $u \in \mathcal{I}(F(x), F(y))$. By the definition of the generated ideal and the assumption on $\mathcal{R}$, there exist finite morphisms: $\psi_i = F(\widetilde{\psi}_i) : F(x_i) \to F(y_i) \in \mathcal{R}$, $\upsilon_i \in \mathcal{A}(F(y_i), F(y))$ and $\omega_i \in \mathcal{A}(F(x), F(x_i))$ for $1 \leq i \leq n$ such that $u = \sum_{i=1}^n \upsilon_i \psi_i \omega_i$.

By Lemma 2.10 (1), we may write $\omega_i = \sum_{t=1}^{m'_i} \delta_{g_s,x_i} F(\omega_{i,g_s})$, where $g_1, \cdots, g_{m'_i} \in G$ are distinct and $\omega_{i,g_s} \in \widehat{\mathcal{A}}(x, g_s x_i)$; and $\upsilon_i = \sum_{s=1}^{m''_i} \delta_{h_t,y} F(\upsilon_{i,h_t})$, where $h_1, \cdots, h_{m''_i} \in G$ are distinct and $\upsilon_{i,h_t} \in \widehat{\mathcal{A}}(y_i, h_t y)$. Then

$$\begin{aligned}
\upsilon_i\psi_i\omega_i &= \big(\sum_{s=1}^{m_i''}\delta_{h_t,y}F(\upsilon_{i,h_t})\big)\big(F(\widetilde{\psi}_i)\big)\big(\sum_{t=1}^{m_i'}\delta_{g_s,x_i}F(\omega_{i,g_s})\big)\\
&= \sum_{1\le s\le m_i'';1\le t\le m_i'}\delta_{h_t,y}F(\upsilon_{i,h_t})F(\widetilde{\psi}_i)\delta_{g_s,x_i}F(\omega_{i,g_s})
\end{aligned}$$

By the naturality of functorial isomorphism δ, we have the following commutative diagrams

$$\begin{array}{ccc}
F\circ g_s(x_i) & \xrightarrow{\delta_{g_s,x_i}} & F(x_i)\\
{\scriptstyle F\circ g_s(\widetilde{\psi}_i)}\downarrow & & \downarrow{\scriptstyle F(\widetilde{\psi}_i)}\\
F\circ g_s(y_i) & \xrightarrow{\delta_{g_s,y_i}} & F(y_i)
\end{array}
\qquad
\begin{array}{ccc}
F\circ g_s(y_i) & \xrightarrow{\delta_{g_s,y_i}} & F(y_i)\\
{\scriptstyle F\circ g_s(\upsilon_{i,h_t})}\downarrow & & \downarrow{\scriptstyle F(\upsilon_{i,h_t})}\\
F\circ g_s(h_ty) & \xrightarrow{\delta_{g_s,h_ty}} & F(h_ty)
\end{array}$$

Hence, we have the following equations

$$\begin{aligned}
&\sum_{1\le s\le m_i'';1\le t\le m_i'}\delta_{h_t,y}F(\upsilon_{i,h_t})F(\widetilde{\psi}_i)\delta_{g_s,x_i}F(\omega_{i,g_s})\\
&= \sum_{1\le s\le m_i'';1\le t\le m_i'}\delta_{h_t,y}F(\upsilon_{i,h_t})\delta_{g_s,y_i}F\circ g_s(\widetilde{\psi}_i)F(\omega_{i,g_s}) \quad \text{(by the left square)}\\
&= \sum_{1\le s\le m_i'';1\le t\le m_i'}\delta_{h_t,y}\delta_{g_s,h_ty}F\circ g_s(\upsilon_{i,h_t})F\circ g_s(\widetilde{\psi}_i)F(\omega_{i,g_s}) \quad \text{(by the right square)}\\
&= \sum_{1\le s\le m_i'';1\le t\le m_i'}\delta_{g_sh_t,y}F\circ g_s(\upsilon_{i,h_t})F\circ g_s(\widetilde{\psi}_i)F(\omega_{i,g_s}) \quad \text{(by } F \text{ is } G\text{-stable)}\\
&= \sum_{r=1}^{n'}\delta_{k_r,y}F\big(\sum_{(s,t)\in\Omega_r}k_rh_t^{-1}\upsilon_{i,h_t}k_rh_t^{-1}\widetilde{\psi}_i\omega_{i,k_rh_t^{-1}}\big),
\end{aligned}$$

where $\Omega_r=\{(s,t)|\ 1\le s\le m_i'';1\le t\le m_i';k_r=g_sh_t\}$ and $k_1,\ k_2\cdots,\ k_{n'}$ are distinct. Then, we have that

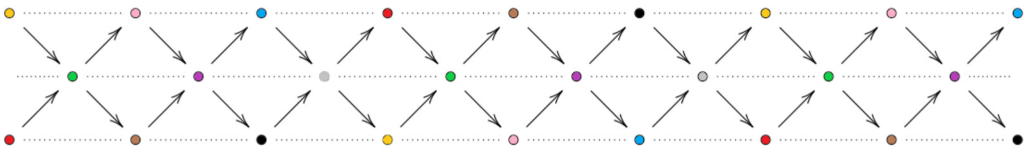
$$\begin{aligned}
u &= \sum_{i=1}^{n}\upsilon_i\psi_i\omega_i\\
&= \sum_{i=1}^{n}\sum_{r=1}^{n'}\delta_{k_r,y}F\big(\sum_{(s,t)\in\Omega_r}k_rh_t^{-1}\upsilon_{i,h_t}k_rh_t^{-1}\widetilde{\psi}_i\omega_{i,k_rh_t^{-1}}\big)\\
&= \sum_{r=1}^{n'}\delta_{k_r,y}F\big(\sum_{i=1}^{n}\sum_{(s,t)\in\Omega_r}k_rh_t^{-1}\upsilon_{i,h_t}k_rh_t^{-1}\widetilde{\psi}_i\omega_{i,k_rh_t^{-1}}\big).
\end{aligned}$$

Let $G_0=\{k_r\}_{r=1}^{n'}\subseteq G$. If we write $u=\sum_{g\in G}\delta_{g,y}\circ F(u_g)$, where $(u_g)_{g\in G}\in\bigoplus_{g\in G}\widehat{\mathcal{A}}(x,gy)$, as $F:\widehat{\mathcal{A}}\to\mathcal{A}$ is a G-precovering, then

$$u_g=\begin{cases}\sum_{i=1}^{n}\sum_{(s,t)\in\Omega_r}k_rh_t^{-1}\upsilon_{i,h_t}k_rh_t^{-1}\widetilde{\psi}_i\omega_{i,k_rh_t^{-1}}, & \text{for } g=k_r\in G_0\\ 0, & \text{for } g\in G\setminus G_0\end{cases}$$

Therefore, $F(u_g)\in\mathcal{I}(F(x),F\circ g(y))$ for any $g\in G$, since $F(\widetilde{\psi}_i)\in\mathcal{I}(F(x_i),F(y_i))$. □

Example 3.5. Let Q be a finite connected acyclic quiver and k be an algebraically closed field. Denote by $\mathcal{D} = \mathcal{D}^b(kQ)$ the bounded derived category of the finite dimensional (left) kQ-modules, [1] the shift functor in $\mathcal{D}$ and τ the AR-translation in $\mathcal{D}$. Then $g = \tau^{-1}[1]$ is an auto-isomorphism of $\mathcal{D}$. Set $G = \langle g \rangle$. The orbit category $\mathcal{C} = \mathcal{D}/G$ is so-called cluster categories. Its objects are the G-orbits of the objects in $\mathcal{D}$. For each $X \in \mathcal{D}_0$, we denote by $\widetilde{X} = (g^i X)_{i\in\mathbb{Z}}$ its G-orbit. See the following diagram, which is an Auslander-Reiten quiver of $\mathcal{D}$



where the same color dots are in same G-orbits. It is well-known that the projection functor $\pi : \mathcal{D} \to \mathcal{C}$ is a G-invariant Galois G-covering, which sends each $X \in \mathcal{D}_0$ to its G-orbit. Let $\mathcal{I} = \mathsf{add}\widetilde{T}$ with $\widetilde{T}$ a cluster tilting object in $\mathcal{C}$. Clearly $\mathcal{I}$ is generated by the set $\mathcal{R}$ consisting of all identity morphisms of objects in $\mathsf{add}\widetilde{T}$. It is easy to see that $\mathcal{R} \subseteq \bigcup_{T_0\in\mathsf{add}T}\{\pi(\mathrm{Hom}_{\mathcal{D}}(T_0, T_0))\}$. Hence, $\mathcal{I}$ is a G-liftable ideal.

Now, let $\widehat{\mathcal{A}}$, $\mathcal{A}$ be linear categories with G a group acting on $\widehat{\mathcal{A}}$, $F : \widehat{\mathcal{A}} \to \mathcal{A}$ be a G-precovering. Assume that $\mathcal{I}$ is a G-liftable ideal on $\mathcal{A}$. Then for any x, $y \in \widehat{\mathcal{A}}_0$ and $g \in G$, we construct the following subgroup of $\widehat{\mathcal{A}}(x, gy)$,

$$\widehat{\mathcal{I}}(x, gy) = \{u \in \widehat{\mathcal{A}}(x, gy) \mid F(u) \in \mathcal{I}(F(x), F \circ g(y))\}.$$

Now, we set

$$\widehat{\mathcal{I}} = \bigcup_{g\in G, x,y\in\widehat{\mathcal{A}}_0} \widehat{\mathcal{I}}(x, gy). \tag{3.1}$$

For any $f \in \widehat{\mathcal{A}}(a, x)$ and $h \in \widehat{\mathcal{A}}(gy, b)$, since $\mathcal{I}$ is an ideal, $F(huf) = F(h)F(u)F(f) \in \mathcal{I}(F(a), F(b))$ for any $u : x \to gy \in \widehat{\mathcal{I}}$. Thus, $huf \in \widehat{\mathcal{I}}(a, b)$. It means that $\widehat{\mathcal{I}}$ is an ideal on $\widehat{\mathcal{A}}$. Since F is a G-precovering and $\mathcal{I}$ is a G-liftable ideal, F induces the following abelian groups isomorphism

$$F_{x,y}|_{\widehat{\mathcal{I}}} : \bigoplus_{g\in G} \widehat{\mathcal{I}}(x, gy) \to \mathcal{I}(F(x), F(y)) : (u_g)_{g\in G} \mapsto \sum_{g\in G} \delta_{g,y} F(u_g).$$

In this case, $F : \widehat{\mathcal{A}} \to \mathcal{A}$ induces a functor $\widetilde{F} : \widehat{\mathcal{A}}/\widehat{\mathcal{I}} \to \mathcal{A}/\mathcal{I}$ and the following exact diagram commutes

$$\begin{array}{ccccc} \widehat{\mathcal{I}} & \rightarrowtail & \widehat{\mathcal{A}} & \xrightarrow{\pi_{\widehat{\mathcal{A}}}} & \widehat{\mathcal{A}}/\widehat{\mathcal{I}} \\ \downarrow & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle \widetilde{F}} \\ \mathcal{I} & \rightarrowtail & \mathcal{A} & \xrightarrow{\pi_{\mathcal{A}}} & \mathcal{A}/\mathcal{I} \end{array} \tag{3.2}$$

where $\pi_{\widehat{\mathcal{A}}}$ and $\pi_{\mathcal{A}}$ are the canonical projection functors.

Moreover, we have the following consequence.

Corollary 3.6. *Let $\widehat{\mathcal{A}}$, $\mathcal{A}$ be linear categories with G a group acting on $\widehat{\mathcal{A}}$. Assume that $F : \widehat{\mathcal{A}} \to \mathcal{A}$ is a G-precovering and $\mathcal{I}$ is ideal of $\mathcal{A}$. Then $\mathcal{I}$ is G-liftable if and only if there is an ideal $\widehat{\mathcal{I}}$ of $\widehat{\mathcal{A}}$ such that F induces the following abelian groups isomorphism*

$$F_{x,y}|_{\widehat{\mathcal{I}}}: \bigoplus_{g\in G}\widehat{\mathcal{I}}(x,gy) \to \mathcal{I}(F(x),F(y)) : (u_g)_g \mapsto \sum_{g\in G}\delta_{g,y}F(u_g), \tag{3.3}$$

for any $x,\ y \in \mathcal{A}_0$.

Denote by $\overline{x}$ the image of object x under the canonical projection functor of linear category. Then the G-action on $\widehat{\mathcal{A}}$ induces the G-action on the quotient category $\widehat{\mathcal{A}}/\widehat{\mathcal{I}}$, defined by $g\overline{x} \triangleq \overline{gx}$ and $g\overline{f} \triangleq \overline{gf}$ for any $x \in \widehat{\mathcal{A}}_0$ and $f \in \mathcal{A}$. Note that F is a G-stable with a G-stabilizer $\delta = (\delta_g)_{g\in G}$, where each $\delta_g : F \circ g \to F$ is a functorial isomorphism. Then for each $g \in G$, there exists a functorial isomorphism $\widetilde{\delta}_g : \widetilde{F} \circ g \to \widetilde{F}$ such that $\widetilde{\delta}_{g,\overline{x}} = \overline{\delta_{g,x}}$ for any $x \in \widehat{\mathcal{A}}_0$. In this case, for any $g,\ h \in G$, and $\overline{x} \in (\widehat{\mathcal{A}}/\widehat{\mathcal{I}})_0$ we have that

$$\widetilde{\delta}_{h,\overline{x}}\widetilde{\delta}_{g,h\overline{x}} = \overline{\delta_{h,x}\delta_{g,hx}} = \overline{\delta_{gh,x}} = \widetilde{\delta}_{gh,\overline{x}}.$$

Thus, $\widetilde{F} : \widehat{\mathcal{A}}/\widehat{\mathcal{I}} \to \mathcal{A}/\mathcal{I}$ is G-stable with a G-stabilizer $\widetilde{\delta} = (\widetilde{\delta}_g)_{g\in G}$. In particular, if F is G-invariant, then so does $\widetilde{F}$. Moreover, if F is dense, for any $\overline{x} \in (\mathcal{A}/\mathcal{I})$, then there exists $x' \in \widehat{\mathcal{A}}_0$, such that $F(x') \cong x$. By the right commutative square of (3.2), $\widetilde{F}(\pi_{\widehat{\mathcal{A}}}(x')) = \pi_{\mathcal{A}}(F(x')) \cong \pi_{\mathcal{A}}(x) = \overline{x}$. It follows that $\widetilde{F}$ is dense. Hence, by the above observations, we have the following result.

Proposition 3.7. *Let $\widehat{\mathcal{A}}$, $\mathcal{A}$ be linear categories with G a group acting on $\widehat{\mathcal{A}}$, $F : \widehat{\mathcal{A}} \to \mathcal{A}$ be a G-precovering. Assume that $\mathcal{I}$ is a G-liftable ideal on $\mathcal{A}$. Then there exists an ideal $\widehat{\mathcal{I}}$ on $\widehat{\mathcal{A}}$ satisfying the isomorphism (3.3) and a G-precovering functor $\widetilde{F} : \widehat{\mathcal{A}}/\widehat{\mathcal{I}} \to \mathcal{A}/\mathcal{I}$, such that the following diagram commutes.*

$$\begin{array}{ccc} \widehat{\mathcal{A}} & \xrightarrow{\pi_{\widehat{\mathcal{A}}}} & \widehat{\mathcal{A}}/\widehat{\mathcal{I}} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle \widetilde{F}} \\ \mathcal{A} & \xrightarrow{\pi_{\mathcal{A}}} & \mathcal{A}/\mathcal{I} \end{array}$$

Moreover, if F is G-covering, then so does $\widetilde{F}$.

Proof. By the above discussions, it suffices to show that $\widetilde{F}_{x,y}$ is an isomorphism. For any $x,\ y \in \widehat{\mathcal{A}}_0$ and $g \in G$, we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigoplus_{g\in G}\widehat{\mathcal{I}}(x,gy) & \longrightarrow & \bigoplus_{g\in G}\widehat{\mathcal{A}}(x,gy) & \longrightarrow & \bigoplus_{g\in G}\widehat{\mathcal{A}}/\widehat{\mathcal{I}}(x,gy) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle F_{x,y}|_{\widehat{\mathcal{I}}}} & & \downarrow{\scriptstyle F_{x,y}} & & \downarrow{\scriptstyle \widetilde{F}_{x,y}} & & \\ 0 & \longrightarrow & \mathcal{I}(F(x),F(y)) & \longrightarrow & \mathcal{A}(F(x),F(y)) & \longrightarrow & \mathcal{A}/\mathcal{I}(F(x),F(y)) & \longrightarrow & 0 \end{array}$$

where the third vertical arrow is induced by $\widetilde{F}$. Because by the construction of $\widehat{\mathcal{I}}$, $F_{x,y}|_{\widehat{\mathcal{I}}}$ is an isomorphism. As $F_{x,y}$ is an isomorphism, so does $\widetilde{F}_{x,y}$. □

Lemma 3.8. *Let $\widehat{\mathcal{A}}$, $\mathcal{A}$ be Hom-finite Krull-Schmidt categories with G a group acting admissibly on $\widehat{\mathcal{A}}$, $F : \widehat{\mathcal{A}} \to \mathcal{A}$ be a G-precovering. Then G-action on $\widehat{\mathcal{A}}/\widehat{\mathcal{I}}$ is also admissible, where $\widehat{I}$ is an ideal of $\widehat{\mathcal{A}}$.*

Proof. Note that $\mathrm{ind}(\widehat{\mathcal{A}}/\widehat{\mathcal{I}}) = \mathrm{ind}(\widehat{\mathcal{A}})\backslash\{x \in \mathrm{ind}(\widehat{\mathcal{A}}) \mid id_x \in \widehat{\mathcal{I}}(x,x)\}$. Assume that $\overline{x}$ is an indecomposable object in $\widehat{\mathcal{A}}/\widehat{\mathcal{I}}$, where $x \in \mathrm{ind}(\widehat{\mathcal{A}})$. For any non-identity $g \in G$, we claim that $g\overline{x} \ncong \overline{x}$. If not, then there exists an isomorphism $\overline{f} : \overline{gx} \to \overline{x}$ with inverse $\overline{h} : \overline{x} \to \overline{gx}$. Thus, $\overline{fh - id_x} = 0$ and $\overline{hf - id_{gx}} = 0$ that is $fh - id_x \in \widehat{I}(x,x)$ and $fh - id_{gx} \in \widehat{I}(gx,gx)$. Note that x is an indecomposable object of $\widehat{\mathcal{A}}$ and so does gx. Since $\widehat{\mathcal{A}}$ is Krull-Schmidt, $\mathsf{End}_{\widehat{\mathcal{A}}}(x)$ is a local R-algebra. Hence, the Jacobson radical $\mathrm{rad}\mathsf{End}_{\widehat{\mathcal{A}}}(x)$ is the unique maximal ideal of $\mathsf{End}_{\widehat{\mathcal{A}}}(x)$. Since $\widehat{I}$ is an ideal, $\widehat{I}(x,x)$ is an ideal of $\mathsf{End}_{\widehat{\mathcal{A}}}(x)$. Then, $\widehat{I}(x,x) \subseteq \mathrm{rad}\mathsf{End}_{\widehat{\mathcal{A}}}(x)$. By the assumption on $\widehat{\mathcal{A}}$, $\widehat{\mathcal{A}}$ is Hom-finite. It follows that $\mathrm{rad}\mathsf{End}_{\widehat{\mathcal{A}}}(x)$ is nilpotent. Then there exists some non-negative integer n such that $(fh - id_x)^n = 0$. Set $k = fh$. Then,

$$\sum_{i=0}^{n}(-1)^i \begin{pmatrix} n \\ i \end{pmatrix} k^{n-i} = 0,$$

where $k^0 = id_x$. Then $f(\sum_{i=0}^{n-1}(-1)^{i-n-1}\binom{n}{i} hk^{n-i-1}) = id_x$. By the similar arguments, we have the equation $(\sum_{i=0}^{n-1}(-1)^{i-n-1}\binom{n}{i} k'^{\,n-i-1}h)f = id_{gx}$ where $k' = hf$. Hence, $f : gx \to x$ is an isomorphism. It leads to a contradiction since the G-action on $\widehat{\mathcal{A}}$ is free. Hence, the G-action on $\widehat{\mathcal{A}}/\widehat{\mathcal{I}}$ is free.

At last, for any $\overline{x}$, $\overline{y} \in \mathrm{ind}(\widehat{\mathcal{A}}/\widehat{\mathcal{I}})$ and $g \in G$, there is a surjective $\widehat{\mathcal{A}}(x, gy) \to \widehat{\mathcal{A}}/\widehat{\mathcal{I}}(\overline{x}, g\overline{y})$. Since G-action on $\widehat{\mathcal{A}}$ is locally bounded, $\widehat{\mathcal{A}}/\widehat{\mathcal{I}}(\overline{x}, g\overline{y}) = 0$ for all but finitely many $g \in G$. It means that the G-action on $\widehat{\mathcal{A}}/\widehat{\mathcal{I}}$ is locally bounded. This completes the proof. □

Theorem 3.9. *Let $\widehat{\mathcal{A}}$, $\mathcal{A}$ be Hom-finite Krull-Schmidt categories with G a group acting admissibly on $\widehat{\mathcal{A}}$, $F : \widehat{\mathcal{A}} \to \mathcal{A}$ be a Galois G-covering. Assume that $\mathcal{I}$ is a G-liftable ideal on $\mathcal{A}$. Then there is a Galois G-covering $\widetilde{F} : \widehat{\mathcal{A}}/\widehat{\mathcal{I}} \to \mathcal{A}/\mathcal{I}$, such that the following diagram commutes,*

$$\begin{array}{ccc} \widehat{\mathcal{A}} & \xrightarrow{\pi_{\widehat{\mathcal{A}}}} & \widehat{\mathcal{A}}/\widehat{\mathcal{I}} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle \widetilde{F}} \\ \mathcal{A} & \xrightarrow{\pi_{\mathcal{A}}} & \mathcal{A}/\mathcal{I} \end{array}$$

where $\widehat{I}$ is an ideal of $\widehat{\mathcal{A}}$ satisfying the isomorphism (3.3).

Proof. By Proposition 3.7 and Lemma 3.8, it is enough to show that $\widetilde{F}$ satisfies the conditions (G1) and (G2) of Definition 2.8.

Assume that $\overline{x}$ is an indecomposable object in $\widehat{\mathcal{A}}/\widehat{\mathcal{I}}$, where $x \in \mathrm{ind}(\widehat{\mathcal{A}})$. For (G1), it suffices to show that $\mathsf{End}_{\mathcal{A}/\mathcal{I}}(\widetilde{F}(\overline{x}))$ is a local R-algebra since $\mathcal{A}/\mathcal{I}$ is Krull-Schmidt. Since $\mathsf{End}_{\mathcal{A}/\mathcal{I}}(\widetilde{F}(\overline{x})) = \frac{\mathrm{Hom}_{\mathcal{A}}(F(x),F(x))}{\mathcal{I}(F(x),F(x))}$, we know that $\mathrm{rad}(\mathsf{End}_{\mathcal{A}/\mathcal{I}}(\widetilde{F}(\overline{x}))) = \frac{\mathrm{rad}(\mathrm{Hom}_{\mathcal{A}}(F(x),F(x)))}{\mathcal{I}(F(x),F(x))}$. It follows that $\mathsf{End}_{\mathcal{A}/\mathcal{I}}(\widetilde{F}(\overline{x}))/\mathrm{rad}(\mathsf{End}_{\mathcal{A}/\mathcal{I}}(\widetilde{F}(\overline{x}))) \cong \mathsf{End}_{\mathcal{A}}(F(x))/\mathrm{rad}(\mathsf{End}_{\mathcal{A}}(F(x)))$ is a division algebra since $F(x)$ is indecomposable and $\mathcal{A}$ is Krull-Schmidt. Thus, $\mathsf{End}_{\mathcal{A}/\mathcal{I}}(\widetilde{F}(\overline{x}))$ is a local algebra.

For (G2), we suppose that $\overline{x}$, $\overline{y} \in \mathrm{ind}(\mathcal{A}/\mathcal{I})$ with $\widetilde{F}(\overline{x}) \cong \widetilde{F}(\overline{y})$, where x, $y \in \mathrm{ind}(\mathcal{A})$. Then there exist isomorphisms $\overline{f} : \overline{F(x)} \to \overline{F(y)}$ and $\overline{h} : \overline{F(y)} \to \overline{F(x)}$ such that $\overline{hf} = \overline{id_{F(x)}}$ and $\overline{fh} = \overline{id_{F(y)}}$. Then, $hf - id_{F(x)} \in I(F(x), F(x))$ and $fh - id_{F(y)} \in I(F(y), F(y))$. Since F satisfies (G1), $F(x)$ and $F(y)$ are indecomposable. Thus, $I(F(x), F(x)) \subsetneq \mathrm{rad}(\mathsf{End}_{\mathcal{A}}(F(x)))$ and $I(F(y), F(y)) \subsetneq \mathrm{rad}(\mathsf{End}_{\mathcal{A}}(F(y)))$. Since $\mathcal{A}$ is Hom-finite, $\mathrm{rad}(\mathsf{End}_{\mathcal{A}}(F(y)))$ and $\mathrm{rad}(\mathsf{End}_{\mathcal{A}}(F(x)))$ are nilpotent. It implies that there exist two non-negative integers n and m, such that $(fh - id_{F(y)})^n = 0$ and $(hf - id_{F(x)})^m = 0$, and consequently, $f : F(x) \to F(y)$ is an isomorphism. Since F satisfies (G2), there exists $g \in G$ such that $y = gx$. In this case, we can see that $\overline{y} = g\overline{x}$. □

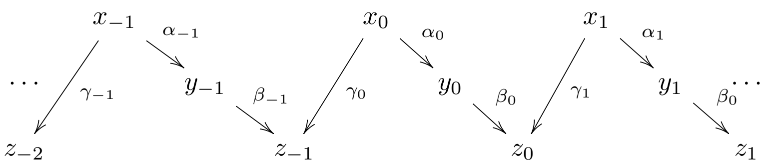
Corollary 3.10 *([3, Proposition 2.3]). Let $\widehat{\mathcal{A}}$, $\mathcal{A}$ be locally bounded categories over a field with G a group acting admissibly on $\widehat{\mathcal{A}}$, $F : \widehat{\mathcal{A}} \to \mathcal{A}$ be a Galois covering. Assume that $\mathcal{I}$ is a F-liftable ideal of $\mathcal{A}$. Then there is a Galois covering $\widetilde{F} : \widehat{\mathcal{A}}/\widehat{\mathcal{I}} \to \mathcal{A}/\mathcal{I}$, where $\widehat{I}$ is same as (3.1).*

Proof. Since $\widehat{\mathcal{A}}$, $\mathcal{A}$ are two locally bounded categories, both $\widehat{\mathcal{A}}$ and $\mathcal{A}$ are Hom-finite Krull-Schmidt linear categories. Since $F : \widehat{\mathcal{A}} \to \mathcal{A}$ is a Galois covering, F is a G-invariant Galois G-covering and the G-action on

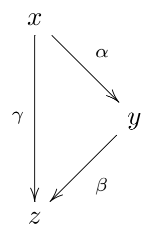
$\widehat{\mathcal{A}}$. In this case, F-liftable ideal $\mathcal{I}$ is just G-liftable ideal. Hence, by Theorem 3.9, there exists a G-invariant Galois G-covering $\widetilde{F} : \widehat{\mathcal{A}}/\widehat{\mathcal{I}} \to \mathcal{A}/\mathcal{I}$. For any $\overline{x} \in \mathrm{ind}(\mathcal{A}/\mathcal{I})$, as $\widetilde{F}$ is dense, there exists $\overline{x'} \in (\widehat{\mathcal{A}}/\widehat{\mathcal{I}})_0$ such that $\widetilde{F}(\overline{x'}) \cong \overline{x}$. Moreover, by Lemma 2.10 (2), $\overline{x'} \in \mathrm{ind}(\widehat{\mathcal{A}}/\widehat{\mathcal{I}})$ and even the fiber $\widetilde{F}^{-1}(\overline{x}) \subseteq \mathrm{ind}(\widehat{\mathcal{A}}/\widehat{\mathcal{I}})$. For any $g \in G$, since $\widetilde{F}$ is G-invariant, $\widetilde{F}(g\overline{x'}) = \widetilde{F}(\overline{x'})$. Hence, the G-orbit $(g\overline{x'})_{g\in G} \subseteq \widetilde{F}^{-1}(\overline{x})$. For any $\overline{y} \in \widetilde{F}^{-1}(\overline{x})$, there exists an isomorphism $\widetilde{F}(\overline{x'}) \cong \widetilde{F}(\overline{y})$. Since $\widetilde{F}$ satisfies (G2), there exists some $g \in G$ such that $\overline{y} = g\overline{x'}$. Thus, the fiber $\widetilde{F}^{-1}(\overline{x})$ is just the G-orbit of $\overline{x}$. It means that G acts transitively of the fiber $\widetilde{F}^{-1}(\overline{x})$. Therefore, $\widetilde{F}$ is a Galois covering. □

Recall that each path algebra kQ over field k can be regard as a k-linear category. Its objects in kQ are just the vertexes of Q, morphisms from vertex x to y are same as paths from x to y, and the composition of any two morphisms is the composition of two paths.

Example 3.11. Let $k\widetilde{Q}$ and kQ be two path algebras over a field k, where the quiver $\widetilde{Q}$ is given by



and Q is given by



Then there is a natural G-action $\rho : \mathbb{Z} \to \mathrm{Aut}(k\widetilde{Q})$ on $k\widetilde{Q}$, given by $\rho(n)(v_i) = v_{i+n}$ and $\rho(n)(f_i) = f_{i+n}$ for any $v_i \in \{x_i, y_i, z_i \mid i \in \mathbb{Z}\}$ and $f_i \in \{\alpha_i, \beta_i, \gamma_i, \alpha_i\beta_i \mid i \in \mathbb{Z}\}$. We define a linear functor $\pi : k\widetilde{Q} \to kQ$ as follows. For each $v_i \in \{x_i, y_i, z_i \mid i \in \mathbb{Z}\}$ and $f_i \in \{\alpha_i, \beta_i, \gamma_i, \alpha_i\beta_i \mid i \in \mathbb{Z}\}$, $\pi(v_i) = v$ and $\pi(f_i) = f$ where $v \in \{x, y, z\}$ and $f \in \{\alpha, \beta, \gamma, \alpha\beta\}$. It is easy to see that π is a Galois G-covering. Consider I as a two-sided ideal of kQ, for any pair a and $b \in \{x, y, z\}$

$$I(a,b) = \begin{cases} k\langle\alpha\beta\rangle, & a = x \text{ and } \ b = z; \\ 0, & \text{otherwise.} \end{cases}$$

Then $\alpha\beta$ is the generator of I. Obviously, $\alpha\beta \in \bigcup_{x_i,z_i} \pi(\mathrm{Hom}_{k\widetilde{Q}}(x_i, z_i))$. Thus, I is a G-liftable ideal of kQ. Moreover,

$$\widetilde{I}(a_i,b_i) = \begin{cases} k\langle\alpha_i\beta_i\rangle, & a_i = x_i \text{ and } \ b_i = z_i; \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that π induces the isomorphism $\pi_{a_i,b_i}|_{\widetilde{I}} : \bigoplus_{g\in\mathbb{Z}} \widetilde{I}(a_i, gb_i) \to I(\pi(a_i), \pi(b_i))$. Thus, we have the Galois G-covering $\widetilde{\pi} : k\widetilde{Q}/\widetilde{I} \to kQ/I$.

4. Admissible ideals

Let $\mathcal{A}$ be an additive category. Set $f : x \to y$ be a morphism in $\mathcal{A}$. Recall that f is said to be *left minimal* if every factorization $f = hf$ implies that h is an automorphism of X; and *left almost split* if f is not a section and for every non-section morphism $g : x \to z$, there is a morphism $g' : y \to z$ such that $g'f = g$. A morphism $f : x \to y$ is called a *source morphism* if f is both left minimal and left almost split. Dually, one can define the notions of *right minimal*, *right almost split* and *sink morphism*. Recall that $f : x \to y$ is irreducible if f is neither a section nor a retraction while every factorization $f = gh$ implies that g is a section or h is a retraction.

A *pseudo-cokernel* of f is a morphism $g : y \to z$ such that for all $a \in \mathcal{A}$ the sequence of abelian groups

$$\mathcal{A}(z, a) \xrightarrow{g^*} \mathcal{A}(y, a) \xrightarrow{f^*} \mathcal{A}(x, a)$$

is exact. The concept of *pseudo-kernel* is defined dually. A *short sequence* in $\mathcal{A}$ is a sequence of two morphisms

$$\eta : x \xrightarrow{u} y \xrightarrow{v} z$$

which is called *pseudo-exact* if u is a pseudo-kernel of v, while v is a pseudo-cokernel of u. The pseudo-exact sequence η is said to be an *almost split sequence* if u is a source morphism and v is a sink morphism.

In [19], under the settings of Hom-finite and Krull-Schmidt, Liu introduces the notions of admissible ideal, pseudo-projective (pseudo-injective) and Auslander-Reiten category. Here, in this section, we modify his definitions without the Hom-finite restriction.

Definition 4.1 *([19]).* Let $\mathcal{A}$ be a Krull-Schmidt category. An ideal $\mathcal{I}$ of $\mathcal{A}$ is called *admissible* provided that it satisfies the following conditions.

(1) If x, y are indecomposable objects in $\mathcal{A}$ with $id_x \notin \mathcal{I}(x, x)$ and $id_y \notin \mathcal{I}(y, y)$, then $\mathcal{I}(x, y) \subsetneq \mathrm{rad}^2(x, y)$.
(2) If $f : x \to y$ is a source morphism in $\mathcal{A}$ with $id_x \notin \mathcal{I}(x, x)$, then every $g \in \mathcal{I}(x, z)$ can be written as $g = hf$ with $h \in \mathcal{I}(y, z)$.
(3) If $f : x \to y$ is a sink morphism in $\mathcal{A}$ with $id_y \notin \mathcal{I}(y, y)$, then every $g \in \mathcal{I}(z, y)$ can be written as $g = fh$ with $h \in \mathcal{I}(z, x)$.

Example 4.2 *([19]).* Let $\mathcal{A}$ be a Hom-finite Krull-Schmidt category.

(1) The infinite radical $\mathrm{rad}^\infty(\mathcal{A}) = \bigcap_{n\geq 1} \mathrm{rad}^n(\mathcal{A})$ of $\mathcal{A}$ is an admissible ideal.
(2) If $\mathcal{B}$ is a subcategory of $\mathcal{A}$ closed under summands, then the ideal of the morphisms factoring through objects in $\mathcal{B}$ is admissible.

Proposition 4.3. *Let $\widehat{\mathcal{A}}$, $\mathcal{A}$ be Krull-Schmidt categories with G a group acting admissibly on $\widehat{\mathcal{A}}$, $F : \widehat{\mathcal{A}} \to \mathcal{A}$ be a Galois G-covering. Assume that $\mathcal{I}$ is a G-liftable ideal on $\mathcal{A}$. Then $\mathcal{I}$ is admissible if and only if $\widehat{\mathcal{I}}$ is admissible.*

Proof. Necessity. Let $\mathcal{I}$ be an admissible ideal on $\mathcal{A}$. For any indecomposable objects x, $y \in \widehat{\mathcal{A}}$, we assume that $id_x \notin \widehat{\mathcal{I}}(x, x)$ and $id_y \notin \widehat{\mathcal{I}}(y, y)$. Then $id_{F(x)} \notin \mathcal{I}(F(x), F(x))$ and $id_{F(y)} \notin \mathcal{I}(F(y), F(y))$. Thus, for any $u \in \widehat{\mathcal{I}}(x, y)$, $F(u) \in \mathcal{I}(F(x), F(y)) \subseteq \mathrm{rad}^2_{\mathcal{A}}(F(x), F(y))$ since $\mathcal{I}$ is an admissible ideal. By Proposition 2.13 (2), we know that $u \in \mathrm{rad}^2_{\widehat{\mathcal{A}}}(x, y)$.

Assume that $f : x \to y$ is a source morphism in $\widehat{\mathcal{A}}$ with $id_x \notin \widehat{\mathcal{I}}(x, x)$. Then $id_{F(x)} \notin \mathcal{I}(F(x), F(x))$. Moreover, by [6, Proposition 3.5], $F(f)$ is a source morphism in $\mathcal{A}$. Then for any $u \in \widehat{\mathcal{I}}(x, z)$, $F(u) \in \mathcal{I}(F(x), F(z))$ can be written as $F(u) = hF(f)$ with $h \in \mathcal{I}(F(y), F(z))$. That is $F(u)$ factorizes through $F(f)$. We may write $h = \sum_{i=1}^{n} \delta_{g_i,z} F(h_i)$, where $g_1, g_2, \cdots, g_n$ are distinct, and $h_i \in \widehat{\mathcal{I}}(y, g_i z)$. By Lemma 2.12, u factorizes through f by some h_i.

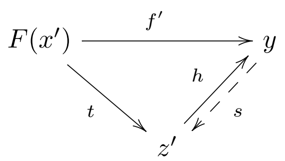
Similarly, one can check the condition (3) in Definition 4.1.

Sufficiency. Assume that $\widehat{\mathcal{I}}$ is an admissible ideal on $\widehat{\mathcal{A}}$. Let x, y be two indecomposable objects of $\mathcal{A}$ with $id_x \notin \mathcal{I}(x,x)$ and $id_y \notin \mathcal{I}(y,y)$. Since F is dense, there exist indecomposable objects x' and $y' \in \widehat{\mathcal{A}}$ such that $x \cong F(x')$ and $y \cong F(y')$. In this case, $\mathcal{I}(x,x) \cong \mathcal{I}(F(x'),F(x'))$ and $\mathcal{I}(y,y) \cong \mathcal{I}(F(y'),F(y'))$. It is easy to see that $id_{x'} \notin \widehat{\mathcal{I}}(x',x')$ and $id_{\sigma y'} \notin \widehat{\mathcal{I}}(\sigma y', \sigma y')$ for any $\sigma \in G$. Since $\widehat{\mathcal{I}}$ is an admissible ideal, $\widehat{\mathcal{I}}(x', \sigma y') \subseteq \operatorname{rad}^2_{\widehat{\mathcal{A}}}(x', \sigma y')$, for any $\sigma \in G$. Since $\mathcal{I}$ is a G-liftable ideal on $\mathcal{A}$, for any $u \in \mathcal{I}(F(x'),F(y'))$, we write $u = \sum_{\sigma\in G} \delta_{\sigma,y} \circ F(u_\sigma)$, where $u_\sigma \in \widehat{\mathcal{I}}(x', \sigma y')$ such that $u_\sigma = 0$ for all but finitely many $\sigma \in G$. Then, $u_\sigma \in \operatorname{rad}^2_{\widehat{\mathcal{A}}}(x', \sigma y')$ and so, $u \in \operatorname{rad}^2_{\mathcal{A}}(F(x'),F(y'))$ by Proposition 2.13 (2).

Suppose that $f : x \to y$ is a source morphism in $\mathcal{A}$ with $id_x \notin \mathcal{I}(x,x)$. Then x is indecomposable. Since F is dense, there exists an indecomposable object $x' \in \widehat{\mathcal{A}}$ such that $x \cong F(x')$. It is easy to see that $id_{x'} \notin \widehat{\mathcal{I}}(x',x')$. Set $v \in \mathcal{I}(F(x'),F(z))$ for any $z \in \widehat{\mathcal{A}}$. We write $v = \sum_{\sigma\in G} \delta_{\sigma,z} \circ F(v_\sigma)$, where $v_\sigma \in \widehat{\mathcal{I}}(x', \sigma z)$ such that $v_\sigma = 0$ for all but finitely many $\sigma \in G$.

If $y = 0$, then $f = 0$. Then $f = F(0)\theta$, where $0 : x' \to 0$ and $\theta : x \to F(x')$ is an isomorphism. Since f is a source map, $0 : x' \to 0$ is a source map by [6, Proposition 3.5]. Thus, for any morphism $\phi \in \widehat{\mathcal{I}}(x', \sigma z)$, ϕ factorizes through $0 : x' \to 0$ since $\widehat{\mathcal{I}}$ is admissible. It follows that v_σ factorizes through $0 : x' \to 0$ and so, $v_\sigma = 0$. Then, $v = 0$ and hence, v factorizes through f.

Otherwise, suppose that $y \neq 0$. Set $f' = f\theta^{-1}$ where $\theta : x \to F(x')$ is an isomorphism. Then f' is a source map. We claim that the source map $f' : F(x') \to y$ is irreducible. Clearly, it is neither a section nor a retraction. Assume that $f' = ht$, where $h : z' \to y$ and $t : F(x') \to z'$ with t is not a section. Since f' is left almost split, there exists a morphism $s : y \to z'$ such that the following diagram commutates.



Then $f' = hsf'$. Since $y \neq 0$ and f' is left minimal, hs is a nonzero automorphism of y. Hence, h is a retraction. It follows that f' is irreducible.

By [6, Proposition 3.4], there is an irreducible morphism $u : x' \to y'$ such that $F(y') \cong y$. From [6, Proposition 3.3], $F(u) : F(x') \to F(y')$ is an irreducible morphism. Since f' is a source map and $F(u)$ is not a section, there is a morphism $w : y \to F(y')$ such that $F(u) = wf'$. Note that f' is not a section. Thus, w is a retraction. Since $\mathcal{A}$ is Krull-Schmidt category, w is an isomorphism by Lemma 2.1. It implies that $F(u)$ is a source map. From [6, Proposition 3.5], u is a source map.

Since $\widehat{\mathcal{I}}$ is admissible, $v_\sigma \in \widehat{\mathcal{I}}(x', \sigma z)$ factorizes through u. That is, for each $\sigma \in G$, there is a morphism h_σ such that $v_\sigma = h_\sigma u$, where $h_\sigma \in \widehat{\mathcal{I}}(y', \sigma z)$. Thus, we have that

$$\begin{aligned} v &= \sum_{\sigma\in G} \delta_{\sigma,z} \circ F(v_\sigma) \\ &= \sum_{\sigma\in G} \delta_{\sigma,z} \circ F(h_\sigma)F(u) \\ &= (\sum_{\sigma\in G} \delta_{\sigma,z} \circ F(h_\sigma))F(u) \\ &= (\sum_{\sigma\in G} \delta_{\sigma,z} \circ F(h_\sigma))wf' \\ &= (\sum_{\sigma\in G} \delta_{\sigma,z} \circ F(h_\sigma))wf\theta^{-1}. \end{aligned}$$

Thus, each morphism in $\mathcal{I}(x,z)$ factorizes through f and $\sum_{\sigma\in G}\delta_{\sigma,z}\circ F(h_\sigma)\in\mathcal{I}$.

By the similar arguments, we can prove that $\mathcal{I}$ satisfies condition (3) of in Definition 4.1. □

Definition 4.4 *([19]).* Let $\mathcal{A}$ be a Krull-Schmidt category. An object $x\in\mathcal{A}$ is called *pseudo-projective* if there exists a sink monomorphism $w\to x$, and dually, *pseudo-injective* if there exists a source epimorphism $x\to v$.

Remark 4.5. In fact, if $\mathcal{A}$ is an abelian category, then the pseudo-projective (pseudo-injective) object coincides with the indecomposable projective (injective) with a unique maximal subobject (quotient object), see [19, Proposition 2.4].

Lemma 4.6. *Let $\widehat{\mathcal{A}}$, $\mathcal{A}$ be linear categories with G a group acting on $\widehat{\mathcal{A}}$, the functor $F:\widehat{\mathcal{A}}\to\mathcal{A}$ be a G-covering. Then $u:x\to y$ is an epimorphism (resp. a monomorphism) if and only if $F(u)$ is an epimorphism (resp. a monomorphism).*

Proof. Necessity. Assume that u is an epimorphism. Suppose that $hF(u)=0$ for $h:F(y)\to z$. Since F is dense, there exists $z'\in\widehat{\mathcal{A}}$ such that $z\cong F(z')$. Let $\alpha:z\to F(z')$ be the isomorphism. Then $\alpha h\in\mathcal{A}(F(y),F(z'))$. By Lemma 2.10 (1), we may write $\alpha h=\sum_{i=1}^{n}\delta_{g_i,z'}F(h_{g_i})$, where $g_1,g_2,\cdots,g_n$ are distinct, and $h_{g_i}\in\widehat{\mathcal{A}}(y,g_iz')$. Then $0=\alpha hF(u)=\sum_{g_i\in G}^{n}\delta_{g_i,z'}F(h_{g_i})F(u)=\sum_{i=1}^{n}\delta_{g_i,z'}F(h_{g_i}u)$. Thus, for any $1\le i\le n$, we have $\delta_{g_i,z'}F(h_{g_i}u)=0$, and hence, $h_{g_i}u=0$ since F is faithful. Since u is an epimorphism, $h_g=0$. Hence, $\alpha h=0$ and so, $h=0$. It implies that $F(u)$ is an epimorphism.

Sufficient. Assume that $F(u)$ is an epimorphism. Let $hu=0$ for $h\in\widehat{\mathcal{A}}(y,Z)$. Then $F(h)F(u)=0$ and hence, $F(h)=0$. Thus, $h=0$ since F is faithful. It means that u is an epimorphism.

Similarly, one can prove the dual statement. □

Proposition 4.7. *Let $\widehat{\mathcal{A}}$, $\mathcal{A}$ be Krull-Schmidt categories with G a group acting admissibly on $\widehat{\mathcal{A}}$, $F:\widehat{\mathcal{A}}\to\mathcal{A}$ be a Galois G-covering. Then an object x in $\widehat{\mathcal{A}}$ is pseudo-projective or pseudo-injective if and only if $F(x)$ is pseudo-projective or pseudo-injective in $\mathcal{A}$.*

Proof. Necessity. If x in $\widehat{\mathcal{A}}$ is pseudo-projective, then there is a sink monomorphism $u:w\to x$. Thus, by [6, Proposition 3.5] and Lemma 4.6, $F(u)$ is a sink monomorphism. Thus, $F(x)$ is pseudo-projective

Sufficiency. Suppose $F(x)$ is pseudo-projective in $\mathcal{A}$. Then there is a sink monomorphism $u:w\to F(x)$. If $w=0$, then $F(0)=0=u$, where $0:0\to x$. By [6, Proposition 3.5] and Lemma 4.6, $0:0\to x$ is a sink monomorphism. If $w\neq 0$, then u is an irreducible map by [23, Proposition 3.14]. From [6, Proposition 3.4], there is an irreducible morphism $u':w'\to x$ such that $F(w')\cong w$. Then $F(u')$ is an irreducible morphism by [6, Proposition 3.3]. Let $\alpha:w\to F(w')$ be an isomorphism. Note that $F(u')\alpha$ is not a retraction. Thus, there is a morphism $\rho:w\to w$ such that $F(u')\alpha=u\rho$. Since $F(u')\alpha$ is an irreducible morphism and u is not a retraction, ρ is a section. Since $\widehat{\mathcal{A}}$ is Krull-Schmidt category, ρ is an isomorphism by Lemma 2.1. Therefore, $F(u')$ is a sink monomorphism since u is a sink monomorphism. It follows that $u':w'\to x$ is a sink monomorphism by [6, Proposition 3.5] and Lemma 4.6. Therefore, x is pseudo-projective. □

We recall that a triangulated category $\mathcal{C}$ is said to be *triangle-connected* if it can not be decomposed as a product of two non-zero triangulated categories; and *triangle-simple* if it admits exactly one indecomposable object up to isomorphism and shift, and the non-zero morphisms between indecomposable objects are isomorphisms. For example, the bounded derived category $\mathcal{D}^b(\mathsf{mod}k)$ is triangle-simple, where k is a field.

Corollary 4.8. *Let $\widehat{\mathcal{A}}$, $\mathcal{A}$ be Hom-finite Krull-Schmidt triangle-connected triangulated categories with G a group acting admissibly on $\widehat{\mathcal{A}}$, $F:\widehat{\mathcal{A}}\to\mathcal{A}$ be a Galois G-covering. Then $\widehat{\mathcal{A}}$ is triangle-simple if and only if $\mathcal{A}$ is triangle-simple.*

Proof. It comes from [19, Proposition 6.2] and Proposition 4.7. □

Definition 4.9 *([19,23]).* Let $\mathcal{A}$ be a Krull-Schmidt category. We call $\mathcal{A}$ a left Auslander-Reiten category if, for every indecomposable $z \in \mathcal{A}$, either z is pseudo-projective or it is the last term of an almost split sequence in $\mathcal{A}$. Dually, $\mathcal{A}$ is a right Auslander-Reiten category if, for every indecomposable $x \in \mathcal{A}$, either x is pseudo-injective or it is the first term of an almost split sequence. If $\mathcal{A}$ is both a left and right Auslander-Reiten category, then we simply call $\mathcal{A}$ an Auslander-Reiten category.

Example 4.10 *([19]).*

(1) If A be an artin algebra then mod A, the category of finitely generated right A-modules, is an Auslander-Reiten category.
(2) If k is an algebraically closed field, then the category of coherent sheaves over $\mathbf{P}^n(k)$ with $n > 1$ can be exhausted an ascending chain of left Auslander-Reiten categories.
(3) The finite dimensional representation of the infinite quiver

$$\cdots \to n \to \cdots \to 2 \to 1$$

over a field from a right Auslander-Reiten category.

Corollary 4.11. *Let $\widehat{\mathcal{A}}$, $\mathcal{A}$ be Krull-Schmidt categories with G a group acting admissibly on $\widehat{\mathcal{A}}$, $F : \widehat{\mathcal{A}} \to \mathcal{A}$ be a Galois G-covering. Then $\widehat{\mathcal{A}}$ is a (left, right) Auslander-Reiten category if and only if $\mathcal{A}$ is a (left, right) Auslander-Reiten category.*

Proof. For the necessity, we suppose that x is an indecomposable object of $\mathcal{A}$. Since F is dense, there exists an indecomposable object $x' \in \widehat{\mathcal{A}}$, such that $F(x') \cong x$. By the assumption, if $\widehat{\mathcal{A}}$ is a left Auslander-Reiten category, then x' is pseudo-projective or it is the last term of an almost split sequence in $\widehat{\mathcal{A}}$. Then by Proposition 4.7 and [6, Theorem 3.7 (2)], we know that $F(x')$ is pseudo-projective or it is the last term of an Auslander-Reiten sequence in $\mathcal{A}$. Thus, $\mathcal{A}$ is a left Auslander-Reiten category. Similarly, we can prove that $\mathcal{A}$ is a right Auslander-Reiten category if $\widehat{\mathcal{A}}$ is a right Auslander-Reiten category.

For the sufficiency, since F is a Galois G-covering, F sends indecomposable objects to indecomposable objects. The rest part of proof comes from the sufficient of Proposition 4.7 and [6, Theorem 3.7 (2)]. □

Corollary 4.12. *Let $\widehat{\mathcal{A}}$, $\mathcal{A}$ be Hom-finite Krull-Schmidt categories with G a group acting admissibly on $\widehat{\mathcal{A}}$, $F : \widehat{\mathcal{A}} \to \mathcal{A}$ be a Galois G-covering. Assume that $\mathcal{I}$ is a G-liftable admissible ideal on $\mathcal{A}$, considering the following statements.*

(1) *$\widehat{\mathcal{A}}$ is a (left, right) Auslander-Reiten category.*
(2) *$\mathcal{A}$ is a (left, right) Auslander-Reiten category.*
(3) *$\widehat{\mathcal{A}}/\widehat{\mathcal{I}}$ is a (left, right) Auslander-Reiten category.*
(4) *$\mathcal{A}/\mathcal{I}$ is a (left, right) Auslander-Reiten category.*

If (1) *or* (2) *holds, then the other statements hold.*

Proof. It follows from Proposition 4.3, Corollary 4.11, Corollary 4.13 and [19, Proposition 2.9]. □

Let k be an algebraically closed field. We denote $D = \mathrm{Hom}_k(-, k) : \mathsf{mod}k \to \mathsf{mod}k$ by the standard k-duality. Let $\mathcal{C}$ be a Hom-finite Krull-Schmidt k-linear triangulated category. Recall from [21] that a *right Serre functor* is an additive functor $S : \mathcal{C} \to \mathcal{C}$ together with a natural isomorphism $D\mathcal{C}(X, -) \cong \mathcal{C}(-, SX)$

for any $X \in \mathcal{C}$. A right Serre functor is said to be *Serre* if it is an equivalence. Dually, one can define the left Serre functor.

From [21], we know that $\mathcal{C}$ is a left (right) Auslander-Reiten category if and only if $\mathcal{C}$ admits a right (left) Serre functor. Thus, by Corollary 4.11, we have the following result, which was proved in [6].

Corollary 4.13. *Let $\widehat{\mathcal{A}}$, $\mathcal{A}$ be Hom-finite Krull-Schmidt triangulated categories with G a group acting admissibly on $\widehat{\mathcal{A}}$, $F : \widehat{\mathcal{A}} \to \mathcal{A}$ be a Galois G-covering. Then $\widehat{\mathcal{A}}$ admits a (left, right) Serre functor if and only if $\mathcal{A}$ admits a (left, right) Serre functor.*

5. Application

Let $\mathcal{C}$ be an additive category. We recall from [6] that $\mathcal{C}$ *has direct sums* provided that any set-indexed family of objects in $\mathcal{C}$ has direct sum. Let $\{X_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$, where I is an indexed set. If $\mathcal{C}$ has direct sums, then the direct sum of $\{X_i\}_{i\in I}$ exists with canonical injection $q_j : X_j \to \oplus_{i\in I} X_i$. By definition of the direct sum, there is a unique morphism $p_j : \oplus_{i\in I} X_i \to X_j$, for each $j \in I$, called *pseudo-projection*, such that

$$p_i q_j = \begin{cases} id_{X_i}, & i = j; \\ 0, & i \neq j, \end{cases} \tag{5.1}$$

for all $i, j \in I$. Recall that an object $M \in \mathcal{C}$ is called *essential* in $\oplus_{i\in I} X_i$ provided that for any morphism $f : M \to \oplus_{i\in I} X_i$, that $f = 0$ if and only if $p_j f = 0$ for all $j \in I$. If every object in $\mathcal{C}$ is essential in $\oplus_{i\in I} X_i$, then $\oplus_{i\in I} X_i$ is called an *essential direct sum*. $\mathcal{C}$ *has essential direct sums* if each family of objects has an essential direct sum.

In what follows, we always assume that $\mathcal{A}$ is a locally bounded linear category over an algebraically closed field k. We define the category of left $\mathcal{A}$-modules, denoted by $\mathsf{Mod}\mathcal{A}$, to be the covariant functors category. It is well known that $\mathsf{Mod}\mathcal{A}$ has essential direct sums, see [6, Lemma 1.2].

For any $x \in \mathcal{A}_0$, the representable functor $P[x] = \mathcal{A}(x, -)$ is a projective $\mathcal{A}$-module in $\mathsf{Mod}\mathcal{A}$. Moreover, $\mathsf{Mod}\mathcal{A}$ has enough projective modules that is for any $M \in \mathsf{Mod}\mathcal{A}$, there is an epimorphism $P \to M$, where P is a projective $\mathcal{A}$-module. We say that a left $\mathcal{A}$-module M is *finite dimensional* if $\sum_{x\in\mathcal{A}_0} \dim_k M(x)$ is finite. We denote by $\mathsf{mod}\mathcal{A}$ the full additive subcategory $\mathsf{Mod}\mathcal{A}$ consisting of all finite dimensional modules. It is well-known that both $\mathsf{Mod}\mathcal{A}$ and $\mathsf{mod}\mathcal{A}$ are abelian categories. Moreover, $\mathsf{mod}\mathcal{A}$ is a Hom-finite Krull-Schmidt category. Since $\mathcal{A}$ is locally bounded, each projective $\mathcal{A}$-module $P[x]$ is finite dimensional and each left $\mathcal{A}$-module M in $\mathsf{mod}\mathcal{A}$ has a projective cover $P \to M$, where P is a finite dimensional projective left $\mathcal{A}$-module. It means that each finite dimensional module is *finitely generated*.

Let M be an $\mathcal{A}$-module in $\mathsf{Mod}\mathcal{A}$. We denote by

$$\mathrm{supp} M = \{x \in \mathcal{A}_0 | M(x) \neq 0\},$$

the *support* of M. Let x be an object of $\mathcal{A}$. $\mathcal{A}_x$ denotes the full subcategory of $\mathcal{A}$ formed by the objects of all $\mathrm{supp} M$, where M is indecomposable and $M(x) \neq 0$. A locally bounded k-linear category $\mathcal{A}$ is called *locally support finite* if for every $x \in \mathcal{A}_0$, $\mathcal{A}_x$ is finite.

The full subcategory of $\mathsf{Mod}\mathcal{A}$ consisting of projective objects is denoted by $\mathsf{Prj}\mathcal{A}$. Note that an $\mathcal{A}$-module P is projective if and only if P is isomorphic to a direct summand of a direct sum of representable functors $P[x]$ where $x \in \mathcal{A}_0$. The full subcategory of $\mathsf{Prj}\mathcal{A}$ consisting of finitely generated projective $\mathcal{A}$-modules is denoted by $\mathsf{prj}\mathcal{A}$.

Let G be a group. The G-action on $\mathcal{A}$ induces a G-action on $\mathsf{Mod}\mathcal{A}$. Fix $g \in G$. Regarding g as the automorphism of $\mathcal{A}$, each left $\mathcal{A}$-module M, one can define $g \cdot M = M \circ g^{-1} : \mathcal{A} \to \mathsf{Mod}k$ and for any morphism $f \in \mathrm{Hom}_{\mathcal{A}}(M,N)$, one can define $g \cdot f : g \cdot M \to g \cdot N$ given by $g \cdot f(x) = f(g^{-1}x)$ for any $x \in \mathcal{A}_0$. In particular, $g \cdot P[x] = P[gx]$, for any $x \in \mathcal{A}_0$ and $g \in G$. If the G-action on $\mathcal{A}$ is free, then the G-action on $\mathsf{mod}\mathcal{A}$ is locally bounded, see [6, Lemma 6.2]. In this case, by [6, Proposition 1.3], for any module M and $N \in \mathsf{mod}\mathcal{A}$, there is a canonical morphism

$$\nu_{M,N} : \oplus_{g\in G}\mathsf{mod}\mathcal{A}(M, gN) \to \mathsf{Mod}\mathcal{A}(M, \oplus_{g\in G} gN). \tag{5.2}$$

Such that for any $u_g \in \mathsf{mod}\mathcal{A}(M, gN)$, $\nu_{M,N}(u_g) = q_g \circ u_g$ and for any $f \in \mathsf{Mod}\mathcal{A}(M, \oplus_{g\in G} gN)$, $f = \nu_M((p_g f)_{g\in G})$.

By Bongartz and Gabriel's classical construction in [7], we know that each G-invariant Galois G-covering $\pi : \mathcal{A} \to \mathcal{B}$ induces an adjoint triple $(\pi_\bullet,\ \pi^\bullet,\ \pi_\diamond)$ between $\mathsf{Mod}\mathcal{A}$ and $\mathsf{Mod}\mathcal{B}$. We will describe $(\pi_\bullet,\ \pi^\bullet)$ explicitly.

Now, we assume that the G-action is free. Let $\pi : \mathcal{A} \to \mathcal{B}$ be a G-invariant Galois G-covering. Follows Bongartz and Gabriel's classical construction in [7], we recall the *push-down* functor

$$\pi_\bullet : \mathsf{Mod}\mathcal{A} \to \mathsf{Mod}\mathcal{B}.$$

For any $M \in \mathsf{Mod}\mathcal{A}$, the left $\mathcal{B}$-module $\pi_\bullet(M)$ is defined as follows. For any $b \in \mathcal{B}_0$,

$$\pi_\bullet(M)(b) := \oplus_{a\in\pi^{-1}(b)} M(a),$$

where $\pi^{-1}(b) = \{a \in \mathcal{A}_0 | \pi(a) = b\}$. Let $\alpha : x \to y$ a morphism in $\mathcal{B}$. Since $\pi : \mathcal{A} \to \mathcal{B}$ is a G-invariant Galois G-covering, for any $a \in \pi^{-1}(x)$, there is an isomorphism

$$\oplus_{b\in\pi^{-1}(y)} \mathcal{A}(a,b) \cong \mathcal{B}(x,y)$$

induced by π. For each pair $(a,b) \in \pi^{-1}(x) \times \pi^{-1}(y)$, there is a unique family $\{\alpha_{b,a} : a \to b\}_{b\in\pi^{-1}(y)}$ such that $\sum_{b\in\pi^{-1}(y)} \pi(\alpha_{b,a}) = \alpha$. Then one defines

$$\pi_\bullet(M)(\alpha) := (M(\alpha_{b,a}))_{(b,a)\in\pi^{-1}(y)\times\pi^{-1}(x)} : \oplus_{a\in\pi^{-1}(x)} M(a) \to \oplus_{b\in\pi^{-1}(y)} M(b).$$

For any morphism $f : M \to N$ in $\mathsf{Mod}\mathcal{A}$, one defines $\pi_\bullet(f) : \pi_\bullet(M) \to \pi_\bullet(N)$ as follows

$$\pi_\bullet(f)(b) := \mathrm{diag}\{f(x) | x \in \pi^{-1}(b)\} : \oplus_{x\in\pi^{-1}(b)} M(x) \to \oplus_{x\in\pi^{-1}(b)} N(x).$$

From [7,14] and [6, Lemma 6.3], follows that the Push-down functor $\pi_\bullet : \mathsf{Mod}\mathcal{A} \to \mathsf{Mod}\mathcal{B}$ is exact and admits a G-stabilizer δ. For any $g \in G$ and $M \in \mathsf{Mod}\mathcal{A}$, the functorial isomorphism $\delta_{g,M} : \pi_\bullet(g\cdot M) \to \pi_\bullet(M)$ is defined as follows. For any $b \in \mathcal{B}_0$, one defines

$$\delta_{g,M}(b) := (\varepsilon_{y,x})_{(y,x)\in\pi^{-1}(b)\times\pi^{-1}(b)} : \oplus_{x\in\pi^{-1}(b)} M(g^{-1}x) \to \oplus_{y\in\pi^{-1}(b)} M(y),$$

where $\varepsilon_{y,x} : M(g^{-1}x) \to M(y)$ is a k-linear map such that

$$\varepsilon_{y,x} = \begin{cases} id_y, & g^{-1}x = y; \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, $\pi_\bullet(P[x]) = P[\pi(x)]$ for any $x \in \mathcal{A}_0$.

Now, we recall the *pull-up* functor, denoted by

$$\pi^\bullet : \mathsf{Mod}\mathcal{B} \to \mathsf{Mod}\mathcal{A},$$

which is both an exact functor and a right adjoint functor of $\pi_\bullet$. For any $N \in \mathsf{Mod}\mathcal{B}$, $\pi^\bullet(N) = N \circ \pi$. For any $f : M \to N$ in $\mathsf{Mod}\mathcal{B}$ and $x \in \mathcal{A}_0$, $\pi^\bullet(f)(x) = f(\pi(x))$. By the definition of $\pi^\bullet$ and $\pi : \mathcal{A} \to \mathcal{B}$ is a G-invariant Galois G-covering, we have that $\pi^\bullet(P[b]) = \mathcal{B}(b, \pi(-)) \cong \oplus_{x\in\pi^{-1}(b)}P[x]$ for any $b \in \mathcal{B}_0$. Moreover, $\pi^\bullet(\mathsf{Prj}\mathcal{B}) \subseteq \mathsf{Prj}\mathcal{A}$. Since the G-action on $\mathcal{A}$ is free, Remark 6.3 together with Theorem 6.2 of [2] implies that the pull-up functor $\pi^\bullet$ is fully faithful. Moreover, by [6, Proposition 6.4], there is a natural isomorphism

$$\gamma_Y : \pi^\bullet\pi_\bullet(Y) \to \oplus_{g\in G}gY,$$

for any $Y \in \mathsf{mod}\mathcal{A}$.

Next, we recall the construction of adjoint isomorphisms ϕ and ψ of $(\pi_\bullet, \pi^\bullet)$, where for any $M \in \mathsf{Mod}\mathcal{A}$ and $N \in \mathsf{Mod}\mathcal{B}$, both

$$\phi_{M,N} : \mathsf{Mod}\mathcal{A}(M, \pi^\bullet(N)) \to \mathsf{Mod}\mathcal{B}(\pi_\bullet(M), N) \tag{5.3}$$

is an isomorphism and nature in two variables M, N. For any $u : M \to \pi^\bullet(N)$ and $b \in \mathcal{B}_0$, $\phi_{M,N}(u)(b) := (u(x))_{x\in\pi^{-1}(b)} : \oplus_{x\in\pi^{-1}(b)}M(x) \to N(b)$ and $\phi_{M,N}(u) := (\phi_{M,N}(u)(b))_{b\in\mathcal{B}_0}$.

It is easy to check that for any morphisms $u \in \mathsf{Mod}\mathcal{A}(M, \pi^\bullet(N))$ and $v \in \mathsf{Mod}\mathcal{B}(\pi_\bullet(M), N)$,

$$\phi_{M,N}(u) = \lambda_N \circ \pi_\bullet(u),$$
$$\phi^{-1}_{M,N}(v) = \pi^\bullet(v) \circ \mu_M$$

where $\lambda_N = \phi_{\pi^\bullet(N),N}(id_{\pi^\bullet(N)}) : \pi_\bullet\pi^\bullet(N) \to N$ and $\mu_M = \phi^{-1}_{M,\pi_\bullet(M)}(id_{\pi_\bullet(M)}) : M \to \pi^\bullet\pi_\bullet(M)$ is the counit and unit of $(\pi_\bullet, \pi^\bullet)$, respectively.

Proposition 5.1 *([6, Theorem 6.5], [10]). Let $\mathcal{A}$ and $\mathcal{B}$ be two locally bounded categories with G a group acting freely on $\mathcal{A}$ and $\pi : \mathcal{A} \to \mathcal{B}$ a G-invariant Galois G-covering. Then the following statements hold.*

(1) *The push-down functor $\pi_\bullet : \mathsf{mod}\mathcal{A} \to \mathsf{mod}\mathcal{B}$ is a G-precovering such that for any X, $Y \in \mathsf{mod}\mathcal{A}$,*

$$\pi_{\bullet X,Y} : \oplus_{g\in G}\mathsf{mod}\mathcal{A}(M, gN) \to \mathsf{mod}\mathcal{B}(\pi_\bullet(X), \pi_\bullet(Y)),$$

given by $\pi_{\bullet X,Y} = \phi_{X,\pi_\bullet(Y)} \circ \mathsf{Mod}\mathcal{A}(X, \gamma_Y^{-1}) \circ \nu_{X,Y}$, is an isomorphism.

(2) *If $\mathcal{A}$ is locally support-finite and G is a torsion-free group, then $\pi_\bullet$ is a Galois G-covering.*

Proposition 5.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be two locally bounded categories with G a group acting freely on $\mathcal{A}$ and $\pi : \mathcal{A} \to \mathcal{B}$ a G-invariant Galois G-covering. Then the push-down functor $\pi_\bullet : \mathsf{mod}\mathcal{A} \to \mathsf{mod}\mathcal{B}$ induces the following isomorphism of subgroups*

$$\oplus_{g\in G}\mathsf{prj}\mathcal{A}(X, gY) \cong \mathsf{prj}\mathcal{B}(\pi_\bullet(X), \pi_\bullet(Y)),$$

for any X, $Y \in \mathsf{mod}\mathcal{A}$. In this case, $\mathsf{prj}\mathcal{B}$ is a G-liftable admissible ideal on $\mathsf{mod}\mathcal{B}$.

Proof. We will divide the proof into following steps.

Step 1: The natural isomorphism ν described in (5.2) induces the subgroups isomorphism

$$\oplus_{g\in G}\mathsf{prj}\mathcal{A}(X, gY) \cong \mathsf{Prj}\mathcal{A}(X, \oplus_{g\in G}gY).$$

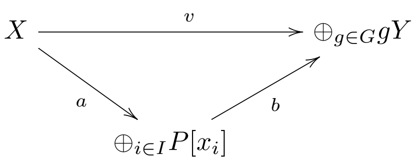
For any $(u_g)_{g\in G} \in \oplus_{g\in G}\mathsf{prj}\mathcal{A}(X, gY)$ with each $u_g \in \mathsf{prj}\mathcal{A}(X, gY)$, since the G-action on $\mathsf{mod}\mathcal{A}$ is locally bounded, there is a finite subset G_0 of G such that $u_h = 0$ for any $h \in G\backslash G_0$. For each $h \in G_0$, since

$u_h \in \mathsf{prj}\mathcal{A}(X, hY)$, there are two morphisms $s_h : X \to P_h$ and $t_h : P_h \to hY$ such that $u_h = t_h \circ s_h$, where $P_h \in \mathsf{prj}\mathcal{A}$. Then $f' = (u_h)_{h\in G_0} = \mathrm{diag}\{t_h|h \in G_0\} \circ (s_h)_{h\in G_0}$ and so $\nu_{X,Y}((u_g)_{g\in G}) = \eta_{G_0} \circ f' = \eta_{G_0} \circ \mathrm{diag}\{t_h|h \in G_0\} \circ (s_h)_{h\in G_0}$, where $(s_h)_{h\in G_0} : X \to \oplus_{h\in G_0} P_h$ is a column-matrix, and $\mathrm{diag}\{t_h|h \in G_0\} : \oplus_{h\in G_0} P_h \to \oplus_{h\in G_0} hY$ is a diagonal matrix. It implies that $\nu_{X,Y}((u_g)_{g\in G}) \in \mathsf{Prj}\mathcal{A}(X, \oplus_{g\in G} gY)$. Since $\nu_{X,Y}$ is injective, the restriction map $\nu_{X,Y}| : \oplus_{g\in G}\mathsf{prj}\mathcal{A}(X, gY) \to \mathsf{Prj}\mathcal{A}(X, \oplus_{g\in G} gY)$ is injective.

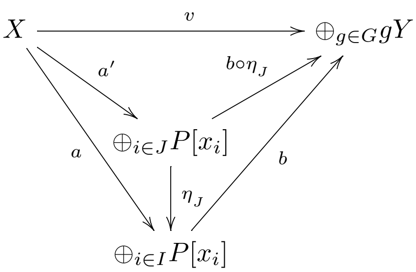
For any $v \in \mathsf{Prj}\mathcal{A}(X, \oplus_{g\in G} gY)$, since ν is an isomorphism and the G-action on $\mathsf{mod}\mathcal{A}$ is locally bounded, there is a finite subset G_0 of G such that for $h \in G_0$, $\nu_{X,Y}((v'_h)_{h\in G_0}) = v$, where $v'_h = p_h \circ v : X \to hY$ for $h \in G_0$.

Next, we shall prove that $v'_h \in \mathsf{prj}\mathcal{A}(X, hY)$ for each $h \in G_0$.

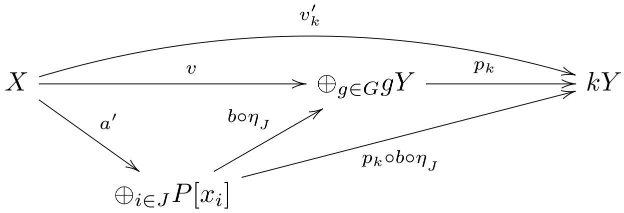
Since $v \in \mathsf{Prj}\mathcal{A}(X, \oplus_{g\in G} gY)$, we assume that there is a projective module $\oplus_{i\in I} P[x_i]$ such that v factors through $\oplus_{i\in I} P[x_i]$. Consider the commutative diagram



For the morphism $a : X \to \oplus_{i\in I} P[x_i]$, since X is finitely generated, there is a finite subset J of I such that there is a commutative diagram



Set $p_k : \oplus_{g\in G} gY \to kY$, for $k \in G_0$. Then, there is a commutative diagram



Thus, $v'_k \in \mathsf{prj}\mathcal{A}(X, kY)$ since $\oplus_{i\in J} P[x_i] \in \mathsf{prj}\mathcal{A}$ for each $k \in G_0$. It implies that $\nu_{X,Y}|$ is surjective and hence, $\nu_{X,Y}|$ is an isomorphism.

Step 2: The adjoint isomorphism $\phi_{M,N}$ described in (5.3) induces the subgroups isomorphism

$$\mathsf{Prj}\mathcal{A}(M, \pi^\bullet(N)) \cong \mathsf{prj}\mathcal{B}(\pi_\bullet(M), N),$$

for any $M, N \in \mathsf{mod}\mathcal{A}$.

For any $f \in \mathsf{Prj}\mathcal{A}(M, \pi^\bullet(N))$, there is a projective $\mathcal{A}$-module P such that f factors through P. Since $\phi_{M,N}(f) = \lambda_N \circ \pi_\bullet(f)$, where λ_N is a counit. Note that $\pi_\bullet$ preserves projective objects. Thus, $\phi_{M,N}(f) = \lambda_N \circ \pi_\bullet(f) \in \mathsf{Prj}\mathcal{B}(\pi_\bullet(M), N)$. Similarly, since $\pi^\bullet$ preserves projective objects, $\phi^{-1}_{M,N}(g) = \pi^\bullet(g) \circ \mu_M \in$

$\mathsf{Prj}\mathcal{A}(M, \pi^\bullet(N))$ for any $g \in \mathsf{Prj}\mathcal{B}(\pi_\bullet(M), N)$, where $\mu_M = \phi^{-1}_{M,\pi_\bullet(M)}(id_{\pi_\bullet(M)})$ is a unit. Then the restriction $\phi_{M,N}|$ of $\phi_{M,N}$ on $\mathsf{Prj}\mathcal{A}(M, \pi^\bullet(N))$ is an isomorphism.

Step 3: Then from Proposition 5.1, we obtain the commutative diagram

$$\begin{array}{ccc} \oplus_{g\in G}\mathsf{prj}\mathcal{A}(X, gY) & \xrightarrow[\cong]{\nu_{X,Y}|} & \mathsf{Prj}\mathcal{A}(X, \oplus_{g\in G} gY) \\ \downarrow{\scriptstyle \pi_{\bullet X,Y}|} & & \cong \downarrow{\scriptstyle (X,\gamma_Y)} \\ \mathsf{prj}\mathcal{B}(\pi_\bullet(X), \pi_\bullet(Y)) & \xrightarrow[\cong]{\phi^{-1}_{X,\pi_\bullet(Y)}|} & \mathsf{prj}\mathcal{B}(X, \pi^\bullet\pi_\bullet(Y)) \end{array}$$

Then, we have that the restriction map $\pi_{\bullet X,Y}|$ is an isomorphism.

Finally, from Example 4.2 and Corollary 3.6, we know that $\mathsf{prj}\mathcal{B}$ is a G-liftable admissible ideal on $\mathsf{mod}\mathcal{B}$. □

Corollary 5.3. [16, Proposition 2.6]*Let $\mathcal{A}$ and $\mathcal{B}$ be two locally bounded categories with G a group acting freely on $\mathcal{A}$ and $\pi : \mathcal{A} \to \mathcal{B}$ a G-invariant Galois G-covering. Then the push-down functor induces a G-precovering $\widetilde{\pi_\bullet} : \underline{\mathsf{mod}}\mathcal{A} \to \underline{\mathsf{mod}}\mathcal{B}$. If $\mathcal{A}$ is locally support-finite and G is a torsion-free group, then $\widetilde{\pi_\bullet}$ is a Galois G-covering.*

Proof. It comes from Proposition 3.7, Proposition 5.1 and Proposition 5.2, Theorem 3.9. □

Next, we consider apply our results into Gorenstein theory. First, we recall some notions.

Let $\mathfrak{U}$ be an abelian category having enough projective objects, denoted by $\mathsf{Prj}\mathfrak{U}$. A complex of projective $P^\bullet : \cdots \to P^{i-1} \to P^i \to P^{i+1} \to \cdots$ is said to be a complete projective complex provided that the complexes $\mathrm{Hom}_{\mathfrak{U}}(P^\bullet, \mathsf{Prj}\mathfrak{U})$ and $\mathrm{Hom}_{\mathfrak{U}}(\mathsf{Prj}\mathfrak{U}, P^\bullet)$ are acyclic. An object X in $\mathcal{A}$ called *Gorenstein projective* if X is a syzygy of a complete projective complex.

Now, we assume that $\mathcal{A}$ is a locally bounded k-linear category. We denote by $\mathcal{GP}(\mathcal{A})$ the full subcategory of $\mathsf{Mod}\mathcal{A}$ consisting of all Gorenstein projective objects in $\mathsf{Mod}\mathcal{A}$. A finitely generated $\mathcal{A}$-module X is called *finitely generated Gorenstein projective*, if X is a syzygy of a complete projective complex of finitely generated projective $\mathcal{A}$-modules. We denote by $\mathcal{G}p(\mathcal{A})$, the full subcategory of $\mathsf{mod}\mathcal{A}$ formed by all finitely generated Gorenstein projective $\mathcal{A}$-modules. It is well-known that $\mathcal{G}p(\mathcal{A})$ is a Frobenious category. By [1, Proposition 4.4], we have that $\mathcal{GP}(\mathcal{A}) \bigcap \mathsf{mod}\mathcal{A} = \mathcal{G}p(\mathcal{A})$. If there is a G-action on $\mathcal{A}$, then $\mathcal{G}p(\mathcal{A})$ is a G-subcategory of $\mathsf{mod}\mathcal{A}$, that is, $g \cdot \mathcal{G}p(\mathcal{A}) \subseteq \mathcal{G}p(\mathcal{A})$ for any $g \in G$.

Corollary 5.4. *Let $\mathcal{A}$ be a locally support-finite with G a torsion-free group acting freely on $\mathcal{A}$, $\mathcal{B}$ be a locally bounded category and $\pi : \mathcal{A} \to \mathcal{B}$ a G-invariant Galois G-covering. Then the triangulated category $\underline{\mathcal{G}p(\mathcal{A})}$ has Serre functor if and only if $\underline{\mathcal{G}p(\mathcal{B})}$ has Serre functor.*

Proof. By [1, Lemma 3.7], we have that the push-down functor $\pi_\bullet$ preserves complete projective complexes of finitely generated projective $\mathcal{A}$-modules. Thus $\pi_\bullet$ sends $\mathcal{G}p(\mathcal{A})$ to $\mathcal{G}p(\mathcal{B})$. Since $\mathcal{G}p(\mathcal{A})$ is a full G-subcategory of $\mathsf{mod}\mathcal{A}$ and $\mathcal{G}p(\mathcal{B})$ is a full subcategory of $\mathsf{mod}\mathcal{B}$, the restriction map $\pi_\bullet|_{\mathcal{G}p} : \mathcal{G}p(\mathcal{A}) \to \mathcal{G}p(\mathcal{B})$ of $\pi_\bullet$ is a G-precovering. Note that $\mathsf{prj} \subseteq \mathcal{G}p$. Thus, by Proposition 5.2, $\mathsf{prj}\mathcal{B}$ is a G-liftable admissible ideal on $\mathcal{G}p(\mathcal{B})$.

Moreover, if $\mathcal{A}$ is locally support-finite and G is a torsion-free group, then $\pi_\bullet$ is a Galois G-covering with an admissible G-action on $\mathsf{mod}\mathcal{A}$. Let X be a finitely generated $\mathcal{B}$-module in $\mathcal{G}p(\mathcal{B})$. From [10], there exists a finitely generated $\mathcal{A}$-module X' such that $X \cong \pi_\bullet(X')$, where X' is a certain direct summand of $\pi^\bullet(X)$. By [1, Lemma 4.2], we have that the pull-up functor $\pi^\bullet$ preserves complete projective complexes. Then $\pi^\bullet(X) \in \mathcal{G}P(\mathcal{A})$. Since $\mathcal{G}P(\mathcal{A})$ is closed under direct summands, we know that $X' \in \mathcal{G}P(\mathcal{A}) \bigcap \mathsf{mod}\mathcal{A}$. Note that $\mathcal{G}P(\mathcal{A}) \bigcap \mathsf{mod}\mathcal{A} = \mathcal{G}p(\mathcal{A})$. Thus, we see that $X' \in \mathcal{G}p(\mathcal{A})$. Therefore, $\pi_\bullet|_{\mathcal{G}p}$ is dense and so, it is a Galois G-covering with an admissible G-action on $\mathcal{G}p(\mathcal{A})$.

By $\mathcal{G}p$ is Hom-finite Krull-Schmidt and Theorem 3.9, $\pi_\bullet|_{\mathcal{G}p}$ induces a Galois G-covering $\overline{\pi_\bullet|_{\mathcal{G}p}} : \underline{\mathcal{G}p(\mathcal{A})} \to \underline{\mathcal{G}p(\mathcal{B})}$.

It is easy to see that $\overline{\pi_\bullet}$ is a triangle functor since $\mathcal{G}p$ is a Frobenious category and $\pi_\bullet$ is exact. Since $\mathcal{G}p$ is Hom-finite Krull-Schmidt, so does $\underline{\mathcal{G}p}$. Then by Corollary 4.13, we have that $\underline{\mathcal{G}p(\mathcal{A})}$ has Serre functor if and only if $\underline{\mathcal{G}p(\mathcal{B})}$ has Serre functor. □

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