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Two terms tilting complexes and Ringel–Hall algebras

YONGGANG HU

Department of Mathematical Sciences, Tsinghua University, Beijing 100084, People's Republic of China
E-mail: huyonggang@emails.bjut.edu.cn

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Abstract. In this paper, we show that under certain mild assumptions, each 2-term silting complex induces isomorphisms between certain subalgebras of Ringel–Hall algebras. This result generalizes the earlier result about classic tilting modules to the silting complexes. Note that 2-term silting complexes are closely related to the support τ-tilting modules. We also give the τ-tilting version of Obul's work.

Keywords. Silting complex; tilting complex; τ-tilting module; Ringel–Hall algebra.

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1. Introduction

In order to answer the question of how deep the connection between quiver representations and Kac–Moody Lie algebras, Ringel [20] introduced the Hall algebra of a finite dimensional algebra over a finite field, which is called Ringel–Hall algebra. It is based on the framework of Steinitz–Hall [8,24] on the Hall algebra of finite abelian p-groups. It is well-known that the Ringel–Hall algebra of a finite dimensional hereditary algebra provides a realization of the positive (negative) part of the corresponding quantum group, see [7,20–22].

The Bernstein–Gelfand–Ponomarev reflection functor is an important functor in the representation theory. It is a special case of tilting functors [2]. Ringel [23] shows that the Bernstein–Gelfand–Ponomarev reflection functors induce isomorphisms between certain subalgebras of Ringel–Hall algebras. Obul [16] generalized the Ringel's work to tilting functors. Later, Obul [17] extended this result into the tilting modules of finite projective dimension in the sense of [15]. Geng and Peng [6] proved that each tilting complex induces the derived equivalence and then an isomorphism between derived Hall algebras.

As a generalization of the classical tilting theory, the concept of silting objects originated from Keller and Vossieck [13]. More recently, Buan and Zhou [3] gave a generalization of the classical tilting theorem, called the silting theorem. They described the relations of torsion pairs between mod A and mod B, where $B = \mathrm{End}_{D^b(A)}(\mathbf{P})$ and $\mathbf{P}$ is a 2-term silting complex in $K^b(\mathrm{proj}\ A)$. It provides us with a basic framework to compute the isomorphisms of subalgebras of Ringel–Hall algebras by the silting theory.

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Now, we present our main result as follows.

Theorem 1.1. *Let k be a fixed finite field with q elements, and we set $\nu = \sqrt{q}$ and $\mathbb{Q}(\nu)$ be the rational function field of ν. Let A be a finite dimensional k-algebra of finite global dimension, $\mathbf{P}$ a 2-term complex in $K^b(\text{proj } A)$ and $B = \text{End}_{K^b(\text{proj } A)}(\mathbf{P})$. If $\mathbf{P}$ is a tilting complex, then the following statements hold*:

(1) The functor $\text{Hom}_{D^b(A)}(\mathbf{P}, -)$ induces the isomorphism between two subalgebras $\mathcal{H}(\mathcal{T}(\mathbf{P}))$ and $\mathcal{H}(\mathcal{Y}(\mathbf{P}))$.
(2) The functor $\text{Hom}_{D^b(A)}(\mathbf{P}, \Sigma(-))$ induces the isomorphism between two subalgebras $\mathcal{H}(\mathcal{F}(\mathbf{P}))$ and $\mathcal{H}(\mathcal{X}(\mathbf{P}))$.

It is well-known that if $\mathbf{P}$ is a 2-term silting complex in $K^b(\text{proj } A)$, then $\text{H}^0(\mathbf{P})$ is a support τ-tilting A-module, which was introduced by Adachi *et al.* [1]. It is remarkable that there is a τ-tilting version of the Brenner–Butler tilting theorem, which was proved by Treffinger [25]. In this paper, we also give the τ-tilting version of Obul's work, see Corollary 3.11.

The paper is organized as follows. In Section 2, we recall some well-known results on the silting theory and the definition of Ringel–Hall algebras. In Section 3, we prove our main results.

2. Preliminaries

Let A be a finite dimensional k-algebra where k is a field. We denote by mod A the category of finitely generated right A-modules. We denote by proj A the full subcategory of mod A generated by the projective modules. Let $D^b(A)$ be the bounded derived category of mod A, with shift functor Σ and $K^b(\text{proj } A)$ the bounded homotopy category of finitely generated projective right A-modules.

A complex $\mathbf{P}$ is said to be of 2-term if $P^i = 0$ for $i \neq -1, 0$. Recall that a 2-term complex $\mathbf{P}$ in $K^b(\text{proj } A)$ is said to be silting if it satisfies the following two conditions:

(1) $\text{Hom}_{D^b(A)}(\mathbf{P}, \Sigma\mathbf{P})$=0;
(2) thick $\mathbf{P}$=$K^b(\text{proj } A)$, where thick $\mathbf{P}$ is the smallest triangulated subcategory closed under direct summands containing $\mathbf{P}$.

In addition, if $\mathbf{P}$ satisfies $\text{Hom}_{D^b(A)}(\mathbf{P}, \Sigma^{-1}\mathbf{P})$=0, then $\mathbf{P}$ is said to be tilting.

Let $\mathbf{P}$ be a 2-term silting complex in $K^b(\text{proj } A)$, and consider the following two full subcategories of mod A,

$$\mathcal{T}(\mathbf{P}) = \{ X \in \text{mod } A \mid \text{Hom}_{D^b(A)}(\mathbf{P}, \Sigma X) = 0\},$$
$$\mathcal{F}(\mathbf{P}) = \{ Y \in \text{mod } A \mid \text{Hom}_{D^b(A)}(\mathbf{P}, Y) = 0\}.$$

Theorem 2.1 [3]. *Let $\mathbf{P}$ be a 2-term silting complex in $K^b(\text{proj } A)$, and $B = \text{End}_{D^b(A)}(\mathbf{P})$. Then the following assertions hold*:

(1) *The pair $(\mathcal{T}(\mathbf{P}), \mathcal{F}(\mathbf{P}))$ is a torsion pair in* mod A.
(2) *There is a triangle*

$$A \to \mathbf{P}' \xrightarrow{f} \mathbf{P}'' \to \Sigma A$$

with $\mathbf{P}', \mathbf{P}''$ *in* $\operatorname{add}\mathbf{P}$.

Consider the 2*-term complex* $\mathbf{Q}$ *in* $K^b(\text{proj } B)$ *induced by the map*

$$\operatorname{Hom}_{D^b(A)}(\mathbf{P}, f) : \operatorname{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}') \to \operatorname{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}'').$$

(3) $\mathbf{Q}$ *is a* 2*-term silting complex in* $K^b(\text{proj } B)$ *such that*

$$\mathcal{T}(\mathbf{Q}) = \mathcal{X}(\mathbf{P}) = \operatorname{Hom}_{D^b(A)}(\mathbf{P}, \Sigma\mathcal{F}(\mathbf{P}))$$
$$\mathcal{F}(\mathbf{Q}) = \mathcal{Y}(\mathbf{P}) = \operatorname{Hom}_{D^b(A)}(\mathbf{P}, \mathcal{T}(\mathbf{P})).$$

(4) *There is an algebra epimorphism* $\Phi_{\mathbf{P}} : A \to \bar{A} = \operatorname{End}_{D^b(B)}(\mathbf{Q})$.

(5) $\Phi_{\mathbf{P}}$ *is an isomorphism if and only if* $\mathbf{P}$ *is tilting.*

(6) *Let* Φ_*: mod $\bar{A} \to$ mod A *be the inclusion functor. Then one obtains the quasi-inverse equivalences between the pair* $(\mathcal{T}(\mathbf{P}), \mathcal{F}(\mathbf{P}))$ *and* $(\mathcal{T}(\mathbf{Q}), \mathcal{F}(\mathbf{Q}))$*. Then*

$$\mathcal{T}(\mathbf{P}) \underset{\Phi_*\operatorname{Hom}_{D^b(B)}(\mathbf{Q},\Sigma -)}{\overset{\operatorname{Hom}_{D^b(A)}(\mathbf{P},-)}{\rightleftarrows}} \mathcal{F}(\mathbf{Q}) ,$$

$$\mathcal{F}(\mathbf{P}) \underset{\Phi_*\operatorname{Hom}_{D^b(B)}(\mathbf{Q},-)}{\overset{\operatorname{Hom}_{D^b(A)}(\mathbf{P},\Sigma -)}{\rightleftarrows}} \mathcal{T}(\mathbf{Q}) .$$

In what follows, the symbol $\mathbf{Q}$ always denotes the induced complex $\mathbf{Q}$. It is a 2-term silting complex in $K^b(\text{proj } B)$ such that the induced pair $(\mathcal{T}(\mathbf{Q}), \mathcal{F}(\mathbf{Q})) = (\mathcal{X}(\mathbf{P}), \mathcal{Y}(\mathbf{P}))$.

Now, we recall the basic definition of Ringel–Hall algebras. Let A be a finite dimensional k-algebra, and let $\operatorname{Iso}(A)$ be the set of isomorphism classes of finite dimensional A-modules. For any finite set $\mathcal{S}$, we denote its cardinality by $|\mathcal{S}|$.

DEFINITION 2.2 [12,19,20]

Let k be a fixed finite field with q elements, and we set $v = \sqrt{q}$ and $\mathbb{Q}(v)$ be the rational function field of v. For any $[M]$, $[N]$, and $[L] \in \operatorname{Iso}(A)$, let $\mathcal{G}_{MN}^L$ be the Hall number defined as (Riedtmann's Formula, see also [18])

$$\mathcal{G}_{MN}^L := \frac{|\operatorname{Aut}_A(L)||\operatorname{Ext}_A^1(M,N)_L|}{|\operatorname{Aut}_A(M)||\operatorname{Aut}_A(N)||\operatorname{Hom}_A(M,N)|},$$

where $\operatorname{Ext}_A^1(M,N)_L$ is the set of all classes of extensions of M by N which are isomorphic to L. The twisted Ringel–Hall algebra $\mathcal{H}(A)$ is a free $\mathbb{Q}(v)$-module with the basis $\{u_{[M]} | [M] \in \operatorname{Iso}(A)\}$ and the multiplication is given by

$$u_{[M]} * u_{[N]} = v^{\langle \mathbf{dim}\, M, \mathbf{dim}\, N\rangle_A} \sum_{[L]\in\operatorname{Iso}(A)} \mathcal{G}_{MN}^L u_{[L]}.$$

Remark 2.3. Let $\mathcal{H}(\mathcal{T}(\mathbf{P}))$, $\mathcal{H}(\mathcal{F}(\mathbf{P}))$ be the $\mathbb{Q}(v)$-submodules with the basis $\{u_{[M]} | M \in \mathcal{T}(\mathbf{P})\}$ and $\{u_{[N]} | N \in \mathcal{F}(\mathbf{P})\}$, respectively. Since the subcategories $\mathcal{T}(\mathbf{P})$, $\mathcal{F}(\mathbf{P})$ are closed under extensions, $\mathcal{H}(\mathcal{T}(\mathbf{P}))$, $\mathcal{H}(\mathcal{F}(\mathbf{P}))$ are subalgebras of $\mathcal{H}(A)$ Similarly, $\mathcal{H}(\mathcal{X}(\mathbf{P}))$, $\mathcal{H}(\mathcal{Y}(\mathbf{P}))$ are subalgebras of $\mathcal{H}(B)$.

3. Main result

The following result was proved in [10], in the setting of abelian categories with arbitrary coproducts. Indeed, it is also true in our case. The proof of the following lemma has contained in [3,11].

Lemma 3.1 [3,10,11]. *For any* $X \in \operatorname{mod} A$, $\operatorname{Hom}_{D^b(A)}(\mathbf{P}, \Sigma^i X) = 0$ *for any* $i < 0$ *and* $i > 1$.

Now, we should recall the definition of the *Grothendieck group* of the finite dimensional k-algebra A. Let F be the free abelian group generated by representatives of the isomorphism classes of objects in mod A. We denote by $[X]$ such a representative. Let F_0 be the subgroup generated by $[X] - [Y] + [Z]$ for all exact sequences $0 \to X \to Y \to Z \to 0$ in mod A. The Grothendieck group $K_0(A)$ is by definition the factor group F/F_0.

PROPOSITION 3.2

Let $\mathbf{P}$ *be a* 2*-term silting complex in* K^b*(proj A) and* $B = End_{D^b(A)}(\mathbf{P})$. *Then the correspondence*

$$\mathbf{dim}\, M \mapsto \sum_{i=0}^{1} (-1)^i \mathbf{dim}\, \operatorname{Hom}_{D^b(A)}(\mathbf{P}, \Sigma^i M),$$

where M is a right A-module, induces the homomorphism $f : K_0(A) \to K_0(B)$. *Similarly, the correspondence*

$$\mathbf{dim}\, N \mapsto \sum_{i=0}^{1} (-1)^i \mathbf{dim}\, \Phi_* \operatorname{Hom}_{D^b(B)}(\mathbf{Q}, \Sigma^i N),$$

where N is a right B-module, induces the homomorphism $g : K_0(B) \to K_0(A)$.

Proof. Since $\mathbf{P}$ is a 2-term silting complex, then by Lemma 3.1, we know that $\operatorname{Hom}_{D^b(A)}(\mathbf{P}, \Sigma^{-1} X) = 0$ and $\operatorname{Hom}_{D^b(A)}(\mathbf{P}, \Sigma^i X) = 0$ for any $X \in$ mod A and $i > 1$. For any short exact sequence $0 \to L \to M \to N \to 0$ in mod A, there exists a distinguished triangle

$$L \to M \to N \to \Sigma L. \tag{1}$$

Applying the functor $\operatorname{Hom}_{D^b(A)}(\mathbf{P}, -)$ to the sequence (1), we have the following long exact sequence in mod B,

$$\begin{aligned} 0 &\to \operatorname{Hom}_{D^b(A)}(\mathbf{P}, L) \to \operatorname{Hom}_{D^b(A)}(\mathbf{P}, M) \to \operatorname{Hom}_{D^b(A)}(\mathbf{P}, N) \\ &\to \operatorname{Hom}_{D^b(A)}(\mathbf{P}, \Sigma L) \to \operatorname{Hom}_{D^b(A)}(\mathbf{P}, \Sigma M) \to \operatorname{Hom}_{D^b(A)}(\mathbf{P}, \Sigma N) \to 0. \end{aligned}$$

Then we have the following equations:

$$\begin{aligned}&\mathbf{dim}\ \mathrm{Hom}_{D^b(A)}(\mathbf{P}, M) - \mathbf{dim}\ \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \Sigma M)\\ &= [\mathbf{dim}\ \mathrm{Hom}_{D^b(A)}(\mathbf{P}, L) - \mathbf{dim}\ \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \Sigma L)]\\ &\quad + [\mathbf{dim}\ \mathrm{Hom}_{D^b(A)}(\mathbf{P}, N) - \mathbf{dim}\ \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \Sigma N)].\end{aligned}$$

Therefore, the given correspondence defines indeed a group homomorphism $K_0(A) \to K_0(B)$. □

Lemma 3.3. *Let M be an arbitrary module in* mod B. *Then the following hold*:

(1) $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \Phi_*\mathrm{Hom}_{D^b(B)}(\mathbf{Q}, M)) = 0$.
(2) $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \Sigma\Phi_*\mathrm{Hom}_{D^b(B)}(\mathbf{Q}, \Sigma M)) = 0$.

Proof. Let

$$0 \to tM \to M \to M/tM \to 0 \tag{2}$$

be the canonical sequence in $(\mathcal{T}(\mathbf{Q}), \mathcal{F}(\mathbf{Q}))$. Applying the functor $\mathrm{Hom}_{D^b(A)}(\mathbf{Q}, -)$ to the sequence (2), by Lemma 3.1 and Φ_* is an exact functor, we have the following long exact sequence:

$$\begin{aligned}0 &\to \Phi_*\mathrm{Hom}_{D^b(B)}(\mathbf{Q}, tM) \to \Phi_*\mathrm{Hom}_{D^b(B)}(\mathbf{Q}, M) \to \Phi_*\mathrm{Hom}_{D^b(B)}(\mathbf{Q}, M/tM)\\ &\to \Phi_*\mathrm{Hom}_{D^b(B)}(\mathbf{Q}, \Sigma tM) \to \Phi_*\mathrm{Hom}_{D^b(B)}(\mathbf{Q}, \Sigma M)\\ &\to \Phi_*\mathrm{Hom}_{D^b(B)}(\mathbf{Q}, \Sigma M/tM) \to 0.\end{aligned}$$

Since $tM \in \mathcal{T}(\mathbf{Q})$ and $M/tM \in \mathcal{F}(\mathbf{Q})$, we obtain isomorphisms

$$\begin{aligned}\Phi_*\mathrm{Hom}_{D^b(B)}(\mathbf{Q}, M) &\cong \Phi_*\mathrm{Hom}_{D^b(B)}(\mathbf{Q}, tM),\\ \Phi_*\mathrm{Hom}_{D^b(B)}(\mathbf{Q}, \Sigma M) &\cong \Phi_*\mathrm{Hom}_{D^b(B)}(\mathbf{Q}, \Sigma M/tM).\end{aligned}$$

By Theorem 2.1(6), we know that

$$\Phi_*\mathrm{Hom}_{D^b(B)}(\mathbf{Q}, tM) \in \mathcal{F}(\mathbf{P}) \text{ and} \Phi_*\mathrm{Hom}_{D^b(B)}(\mathbf{Q}, \Sigma M/tM) \in \mathcal{T}(\mathbf{P}).$$

It implies that

$$\begin{aligned}&\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \Phi_*\mathrm{Hom}_{D^b(B)}(\mathbf{Q}, M)) \cong \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \Phi_*\mathrm{Hom}_{D^b(B)}(\mathbf{Q}, tM)) = 0\\ &\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \Sigma\Phi_*\mathrm{Hom}_{D^b(B)}(\mathbf{Q}, \Sigma M))\\ &\quad \cong \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \Sigma\Phi_*\mathrm{Hom}_{D^b(B)}(\mathbf{Q}, \Sigma M/tM)) = 0.\end{aligned}$$ □

Lemma 3.4. *Let X be an arbitrary module in* mod A. *Then the following hold*:

(1) $\Phi_*\mathrm{Hom}_{D^b(B)}(\mathbf{Q}, \mathrm{Hom}_{D^b(A)}(\mathbf{P}, X)) = 0$.
(2) $\Phi_*\mathrm{Hom}_{D^b(B)}(\mathbf{Q}, \Sigma\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \Sigma X)) = 0$.

Proof. Let

$$0 \to tX \to X \to X/tX \to 0 \tag{3}$$

be the canonical sequence in $(\mathcal{T}(\mathbf{P}), \mathcal{F}(\mathbf{P}))$. Applying the functor $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, -)$ to the sequence (3), by Lemma 3.1, we have the following long exact sequence:

$$\begin{aligned} 0 &\to \mathrm{Hom}_{D^b(A)}(\mathbf{P}, tX) \to \mathrm{Hom}_{D^b(A)}(\mathbf{P}, X) \to \mathrm{Hom}_{D^b(A)}(\mathbf{P}, M/tM) \\ &\to \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \Sigma tX) \to \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \Sigma X) \to \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \Sigma X/tX) \to 0. \end{aligned}$$

Since $tX \in \mathcal{T}(\mathbf{P})$ and $X/tX \in \mathcal{F}(\mathbf{P})$, we obtain isomorphisms

$$\begin{aligned} \mathrm{Hom}_{D^b(A)}(\mathbf{P}, M) &\cong \mathrm{Hom}_{D^b(A)}(\mathbf{P}, tX), \\ \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \Sigma M) &\cong \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \Sigma X/tX). \end{aligned}$$

By Theorem 2.1(6), we know that

$$\mathrm{Hom}_{D^b(B)}(\mathbf{P}, tX) \in \mathcal{F}(\mathbf{Q}) \text{ and } \mathrm{Hom}_{D^b(A)}(\mathbf{Q}, \Sigma X/tX) \in \mathcal{T}(\mathbf{Q}).$$

It implies that

$$\begin{aligned} &\Phi_*\mathrm{Hom}_{D^b(B)}(\mathbf{Q}, \mathrm{Hom}_{D^b(A)}(\mathbf{P}, X)) \cong \Phi_*\mathrm{Hom}_{D^b(B)}(\mathbf{Q}, \mathrm{Hom}_{D^b(A)}(\mathbf{P}, tX)) = 0 \\ &\Phi_*\mathrm{Hom}_{D^b(B)}(\mathbf{Q}, \Sigma\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \Sigma X)) \\ &\qquad \cong \Phi_*\mathrm{Hom}_{D^b(B)}(\mathbf{Q}, \Sigma\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \Sigma X/tX)) = 0. \end{aligned}$$

□

PROPOSITION 3.5

Let $\mathbf{P}$ *be a* 2*-term silting complex in* $K^b(\text{proj } A)$ *and* $B = \mathrm{End}_{D^b(A)}(\mathbf{P})$. *Then the correspondence*

$$\mathbf{dim}\ M \mapsto \sum_{i=0}^{1} (-1)^i \mathbf{dim}\ \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \Sigma^i M),$$

where M *is a right* A*-module, induces the isomorphism* $f : K_0(A) \to K_0(B)$.

Proof. For any simple module S in mod B, it is easy to see that $S \in \mathcal{T}(\mathbf{Q})$ or $S \in \mathcal{F}(\mathbf{Q})$. If $S \in \mathcal{T}(\mathbf{Q})$, then $S \cong \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \Sigma\Phi_*\mathrm{Hom}_{D^b(B)}(\mathbf{Q}, S))$ and by Lemma 3.3(1), we have $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \Phi_*\mathrm{Hom}_{D^b(B)}(\mathbf{Q}, S)) = 0$. In this case, we know that $f(-\mathbf{dim}\ \Phi_*\mathrm{Hom}_{\mathrm{D^b(B)}}(\mathbf{Q}, \mathrm{S})) = \mathbf{dim}\ \mathrm{S}$. If $S \in \mathcal{F}(\mathbf{Q})$, then there is an isomorphism $S \cong \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \Phi_*\mathrm{Hom}_{D^b(B)}(\mathbf{Q}, \Sigma S))$ and by Lemma 3.3(2), we have that $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \Sigma\Phi_*\mathrm{Hom}_{D^b(B)}(\mathbf{Q}, \Sigma S)) = 0$. Therefore, we obtain that $f(\mathbf{dim}\ \Phi_*\mathrm{Hom}_{\mathrm{D^b(B)}}(\mathbf{Q}, \Sigma\mathrm{S})) = \mathbf{dim}\ \mathrm{S}$.

Similarly, by Lemma 3.4, one can prove that $g : K_0(B) \to K_0(A)$ is also epic. Therefore, the ranks of $K_0(A)$ and $K_0(B)$ are equal. □

Let $\mathcal{C}$ be a triangulated category. Let F be the free abelian group generated by representatives of the isomorphism classes of objects in $\mathcal{C}$. We denote by $[X]$ such a representative. Let F_0 be the subgroup generated by $[X]-[Y]+[Z]$ for all triangles $X \to Y \to Z \to \Sigma X$ in $\mathcal{C}$. The *Grothendieck group* $K_0(\mathcal{C})$ is by definition the factor group F/F_0.

COROLLARY 3.6

Let $\mathbf{P}$ *be a* 2*-term silting complex in* $K^b(\text{proj } A)$ *and* $B = \text{End}_{D^b(A)}(\mathbf{P})$. *Then* f *in Proposition* 3.5 *induces the isomorphism* $K_0(D^b(A)) \cong K_0(D^b(B))$.

Proof. From [9, Chapter III, Lemma 1.2], we know that the canonical embedding of mod A into $D^b(A)$ induces an isomorphism of $K_0(A)$ with $K_0(D^b(A))$. Similarly, there is an isomorphism $K_0(B) \cong K_0(D^b(B))$. Thus, from Proposition 3.5, f induces the isomorphism $K_0(D^b(A)) \cong K_0(D^b(B))$. □

Let A be a finite dimensional k-algebra and $\mathbf{X} = (\mathbf{X}^i, d^i)$ be a bounded complex in $D^b(A)$. From [9, Chapter III], the dimension vector of $\mathbf{X}$ is defined as $\mathbf{dim}\ \mathbf{X} = \sum_{\mathrm{i}\in\mathbb{Z}}(-1)^{\mathrm{i}}\mathbf{dim}\ \mathbf{X}^{\mathrm{i}}$. The preceding sum is finite due to our hypothesis on $\mathbf{X}$.

Assume now that A is an algebra of finite global dimension. We recall from [9, Chapter III, Lemma 1.4] that the *Euler characteristic* of $D^b(A)$ is the bilinear form on $K_0(D^b(A))$ defined by

$$\langle \mathbf{dim}\ \mathbf{X}, \mathbf{dim}\ \mathbf{Y}\rangle_A = \sum_{i\in\mathbb{Z}}^{\infty}(-1)^i \dim_k \text{Hom}_{D^b(A)}(\mathbf{X}, \Sigma^i \mathbf{Y}),$$

where $\mathbf{X}$, $\mathbf{Y}$ are complexes in $D^b(A)$. In particular, if $\mathbf{X}$, $\mathbf{Y}$ are 0-stalk complex, then

$$\langle \mathbf{dim}\ \mathbf{X}, \mathbf{dim}\ \mathbf{Y}\rangle_A = \sum_{i=0}^{\infty}(-1)^i \dim_k \text{Ext}_A^i(\mathbf{X}, \mathbf{Y}).$$

PROPOSITION 3.7

Let A *be an algebra of finite global dimension*, $\mathbf{P}$ *a* 2*-term complex in* $K^b(\text{proj } A)$ *and* $B = \text{End}_{D^b(A)}(\mathbf{P})$. *If* $\mathbf{P}$ *is a tilting complex*, *then the map* f *in Proposition* 3.2 *is an isometry of the Euler characteristics of* A *and* B. *That is*, *for any complexes* $\mathbf{X}$ *and* $\mathbf{Y}$, *we have*

$$\langle \mathbf{dim}\ X, \mathbf{dim}\ Y\rangle_A = \langle f(\mathbf{dim}\ X), f(\mathbf{dim}\ Y)\rangle_B.$$

In particular, *for any* A*-modules* M *and* N, *we have*

$$\langle \mathbf{dim}\ M, \mathbf{dim}\ N\rangle_A = \langle f(\mathbf{dim}\ M), f(\mathbf{dim}\ N)\rangle_B.$$

Proof. Assume that $\mathbf{P} : P^{-1} \xrightarrow{d} P^0$, where all P^i are finitely generated projective modules. Let $\mathbf{P}_1, \ldots, \mathbf{P}_n$ denote the pairwise nonisomorphic indecomposable summands of $\mathbf{P}$. We claim that the vectors $\mathbf{dim}\ \mathbf{P}_{\mathrm{i}}$, where $1 \leq i \leq n$, constitute a basis of $K_0(D^b(A))$. Since $K^b(\text{proj } A)$ is a Hom-finite Krull–Schmidt category, $\text{Hom}_{K^b(\text{proj } A)}(\mathbf{P}, -) : \text{add}\,\mathbf{P} \to$

proj B is an equivalence, by [14, Proposition 2.3]. Thus, the B-modules

$$\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}_1), \ldots, \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}_n)$$

form a complete set of representatives of the isomorphism classes of indecomposable projective modules. From [5], B also has finite global dimension. Thus, the Euler characteristics of $K_0(D^b(B))$ is well-defined.

Note that there is a distinguished triangle

$$\mathbf{P}_i^{-1} \xrightarrow{-d} \mathbf{P}_i^0 \to \mathbf{P}_i \to \Sigma\mathbf{P}_i^{-1}. \tag{4}$$

Note that for any $i < -1$ and $j > 1$, $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \Sigma^i\mathbf{P}) = \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \Sigma^j\mathbf{P}) = 0$ since $\mathbf{P}$ is a 2-term silting complex. Applying the functor $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, -)$ to the sequence (4), we have the following exact sequence in mod B,

$$0 \to \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \Sigma^{-1}\mathbf{P}_i) \to \mathrm{Hom}_{D^b(A)}\left(\mathbf{P}, \mathbf{P}_i^{-1}\right) \to \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}_i^0) \to$$
$$\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}_i) \to \mathrm{Hom}_{D^b(A)}\left(\mathbf{P}, \Sigma\mathbf{P}_i^{-1}\right) \to \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \Sigma\mathbf{P}_i^0) \to 0.$$

Thus, we have

$$\begin{aligned}
&\mathbf{dim}\,\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}_i) - \mathbf{dim}\,\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \Sigma^{-1}\mathbf{P}_i)\\
&= \mathbf{dim}\,\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}_i^0) - \mathbf{dim}\,\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \Sigma\mathbf{P}_i^0)\\
&\quad - \mathbf{dim}\,\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}_i^{-1}) + \mathbf{dim}\,\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \Sigma\mathbf{P}_i^{-1}).
\end{aligned}$$

It follows that

$$\begin{aligned}
f(\mathbf{dim}\,\mathbf{P}_i) &= f(\mathbf{dim}\,\mathbf{P}_i^0) - f(\mathbf{dim}\,\mathbf{P}_i^{-1})\\
&= \mathbf{dim}\,\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}_i) - \mathbf{dim}\,\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \Sigma^{-1}\mathbf{P}_i)\\
&= \mathbf{dim}\,\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}_i),
\end{aligned}$$

since $\mathbf{P}$ is a tilting complex.

Then $f(\mathbf{dim}\,\mathbf{P}_i)$, where $1 \leq i \leq n$ constitute a basis of $K_0(D^b(B))$. Because, by Corollary 3.6, f is an isomorphism, this implies our claim.

Also, the projectivity of the B-modules $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}_i)$ and [4, Lemma 3.5] imply that, for any i, j such that $1 \leq i, j \leq n$,

$$\begin{aligned}
\langle f(\mathbf{dim}\,\mathbf{P}_i), f(\mathbf{dim}\,\mathbf{P}_j)\rangle_B &= \langle \mathbf{dim}\,\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}_i), \mathbf{dim}\,\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}_j)\rangle_B\\
&= \dim_k \mathrm{Hom}_B(\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}_i), \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}_j))\\
&= \dim_k \mathrm{Hom}_{D^b(A)}(\mathbf{P}_i, \mathbf{P}_j)\\
&= \langle \mathbf{dim}\,\mathbf{P}_i, \mathbf{dim}\,\mathbf{P}_j\rangle_A.
\end{aligned}$$

It completes the proof. □

Now, we are in position to prove our main results.

Theorem 3.8. *Let k be a fixed finite field with q elements, and we set $\nu = \sqrt{q}$ and $\mathbb{Q}(\nu)$ be the rational function field of ν. Let A be a finite dimensional k-algebra of finite global dimension, $\mathbf{P}$ a 2-term complex in $K^b(\mathrm{proj}\, A)$ and $B = \mathrm{End}_{D^b(A)}(\mathbf{P})$. If $\mathbf{P}$ is a tilting complex, then the following statements hold*:

(1) *The functor $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, -)$ induces the isomorphism between two subalgebras $\mathcal{H}(\mathcal{T}(\mathbf{P}))$ and $\mathcal{H}(\mathcal{Y}(\mathbf{P}))$.*

(2) *The functor $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \Sigma(-))$ induces the isomorphism between two subalgebras $\mathcal{H}(\mathcal{F}(\mathbf{P}))$ and $\mathcal{H}(\mathcal{X}(\mathbf{P}))$.*

Proof.

(1) For convenience, we set $F = \mathrm{Hom}_{D^b(A)}(\mathbf{P}, -) : \mathcal{T}(\mathbf{P}) \to \mathcal{Y}(\mathbf{P})$ and its quasi-inverse $G = \Phi_* \mathrm{Hom}_{D^b(B)}(\mathbf{Q}, \Sigma -)$. We define the linear map $\phi : \mathcal{H}(\mathcal{T}(\mathbf{P})) \to \mathcal{H}(\mathcal{Y}(\mathbf{P}))$ by $\phi(u_{[M]}) = u_{[F(M)]}$ for any $M \in \mathcal{T}(\mathbf{P})$. Clearly, it is well-defined. Since F is an equivalence between $\mathcal{T}(\mathbf{P})$ and $\mathcal{Y}(\mathbf{P})$, ϕ is a bijection. Thus, it suffices to show that ϕ is a homomorphism.

First, we claim that $\mathcal{G}^L_{MN} = \mathcal{G}^{F(L)}_{F(M)F(N)}$ for any $M, N \in \mathcal{T}(\mathbf{P})$.

For any $M, N \in \mod A$, set $\mathcal{E}^L_{MN} = \{(f, g) : 0 \to N \xrightarrow{f} L \xrightarrow{g} M \to 0\}$. By Riedtmann's formula, we have

$$|\mathcal{E}^L_{MN}| = \frac{|\mathrm{Aut}_A(L)||\mathrm{Ext}^1_A(M, N)_L|}{|\mathrm{Hom}_A(M, N)|}.$$

Then $\mathcal{G}^L_{MN} = \frac{|\mathcal{E}^L_{MN}|}{|\mathrm{Aut}_A(M)||\mathrm{Aut}_A(N)|}$. Since F is a quasi-isomorphism on $\mathcal{T}(\mathbf{P})$, thus, $\mathrm{Aut}_A(M) \cong \mathrm{Aut}_B(F(M))$ and $\mathrm{Aut}_A(N) \cong \mathrm{Aut}_B(F(N))$ for any $M, N \in \mathcal{T}(\mathbf{P})$. Hence, it is enough to show that $|\mathcal{E}^L_{MN}| = |\mathcal{E}^{F(L)}_{F(M)F(N)}|$ for any $M, N \in \mathcal{T}(\mathbf{P})$.

For any short exact sequence $0 \to N \xrightarrow{f} L \xrightarrow{g} M \to 0$ in $\mathcal{T}(\mathbf{P})$, we have a distinguished triangle

$$N \xrightarrow{f} L \xrightarrow{g} M \to \Sigma N. \tag{5}$$

Applying the functor F to (5), by Lemma 3.1 and $N \in \mathrm{Ker}\, \mathrm{Hom}_A(\mathbf{P}, \Sigma(-))$, we have the exact sequence

$$0 \to F(N) \xrightarrow{F(f)} F(L) \xrightarrow{F(g)} F(M) \to 0.$$

Hence, F induces a map $\bar{F}$ from $\mathcal{E}^L_{MN}$ to $\mathcal{E}^{F(L)}_{F(M)F(N)}$ given by $\bar{F}((f, g)) = (F(f), F(g))$ for any $(f, g) \in \mathcal{E}^L_{MN}$. For any exact sequence $0 \to F(N) \xrightarrow{F(f)} F(L) \xrightarrow{F(g)} F(M) \to 0$, applying G, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & GF(N) & \xrightarrow{GF(f)} & GF(L) & \xrightarrow{GF(g)} & GF(M) & \longrightarrow & 0 \\
 & & \downarrow{\cong} & & \downarrow{\cong} & & \downarrow{\cong} & & \\
0 & \longrightarrow & N & \xrightarrow{f} & L & \xrightarrow{g} & M & \longrightarrow & 0
\end{array}$$

Thus, $\bar{F}$ is surjective. Similarly, one can prove that G induces a surjective map from $\mathcal{E}_{F(M)F(N)}^{F(L)}$ to $\mathcal{E}_{MN}^{L}$. Thus, $|\mathcal{E}_{MN}^{L}| = |\mathcal{E}_{F(M)F(N)}^{F(L)}|$.

Second, we claim that $\langle \mathbf{dim}\, M, \mathbf{dim}\, N\rangle_A = \langle \mathbf{dim}\, F(M), \mathbf{dim}\, F(N)\rangle_B$ for any M, $N \in \mathcal{T}(\mathbf{P})$.

Indeed, for any $X \in \mathcal{T}(\mathbf{P})$, because $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \Sigma M) = 0$, $f(\mathbf{dim}\, X) = \mathbf{dim}\, F(X)$. Thus, by Proposition 3.7, we have

$$\begin{aligned}\langle \mathbf{dim}\, F(M), \mathbf{dim}\, F(N)\rangle_B &= \langle f(\mathbf{dim}\, M), f(\mathbf{dim}\, N)\rangle_B \\ &= \langle \mathbf{dim}\, M, \mathbf{dim}\, N\rangle_A.\end{aligned}$$

Finally, we check that $\phi(u_{[M]} * u_{[M]}) = \phi(u_{[M]}) * \phi(u_{[M]})$ for any $M, N \in \mathcal{T}(\mathbf{P})$.

By the above results, we obtain that

$$\begin{aligned}\phi(u_{[M]} * u_{[M]}) &= \phi(\nu^{\langle \mathbf{dim}\, M, \mathbf{dim}\, N\rangle_A} \sum_{[L]} \mathcal{G}_{MN}^{L} u_{[L]}) \\ &= \nu^{\langle \mathbf{dim}\, M, \mathbf{dim}\, N\rangle_A} \sum_{[L]} \mathcal{G}_{MN}^{L} \phi(u_{[L]}) \\ &= \nu^{\langle \mathbf{dim}\, M, \mathbf{dim}\, N\rangle_A} \sum_{[L]} \mathcal{G}_{MN}^{L} u_{\phi([L])} \\ &= \nu^{\langle \mathbf{dim}\, F(M), \mathbf{dim}\, F(N)\rangle_B} \sum_{[F(L)]} \mathcal{G}_{F(M)F(N)}^{F(L)} u_{[F(L)]} \\ &= u_{[F(M)]} * u_{[F(N)]} \\ &= \phi(u_{[M]}) * \phi(u_{[M]}).\end{aligned}$$

Therefore, ϕ is an isomorphism. Similarly, one can prove the statement (2). □

At last, by Theorem 3.8, we give the τ-tilting version of [16, Theorem 1 and 2].

Let $\mathbf{P}$ is a 2-term silting complex in $K^b(\mathrm{proj}\, A)$. Recall that $\mathbf{P}$ is said to be separating if the induced torsion pair $(\mathcal{T}(\mathbf{P}), \mathcal{F}(\mathbf{P}))$ in mod A is split. By [3], if $\mathbf{P}$ is separating, then $\mathbf{P}$ is a tilting complex.

Next, we recall some notions in τ-tilting theory, which will be used.

DEFINITION 3.9 [2]

Let M be a module in mod A.

(1) M is called *τ-rigid* if $\mathrm{Hom}_A(M, \tau M) = 0$.
(2) M is called *τ-tilting* if it is τ-rigid and $|M| = |A|$.
(3) M is called *support τ-tilting* if it is a τ-tilting $(A/\langle e\rangle)$-module for some idempotent e of A.

DEFINITION 3.10 [1]

Let (M, P) be a pair with $M \in \mathrm{mod}\ A$ and $P \in \mathrm{proj}\, A$.

(1) We call (M, P) a *τ-rigid* pair if M is τ-rigid and $\mathrm{Hom}_A(P, M) = 0$.
(2) We call (M, P) a *support τ-tilting pair* if (M, P) is τ-rigid and $|M| + |P| = |A|$.

Let M be an object in mod A and $B = \mathrm{End}_A(M)$. We set two classes of modules $\mathcal{T}(M) = \mathrm{Fac}\, M$ and $\mathcal{F}(M) = \mathrm{KerHom}_A(M, -)$. By [1, Proposition 2.16], one can see that for a support τ-tilting module M, there is a torsion pair $(\mathcal{T}(M), \mathcal{F}(M))$. Moreover, there is a torsion pair $(\mathcal{X}(M), \mathcal{Y}(M))$ in mod B induced by the support τ-tilting module M, see [25].

COROLLARY 3.11

Let k be a fixed finite field with q elements, and we set $v = \sqrt{q}$ and $\mathbb{Q}(v)$ be the rational function field of v. Let A be a finite dimensional k-algebra of finite global dimension, and (M, P) a support τ-tilting pair. If $\mathbf{P} = (P_1 \oplus P \xrightarrow{(f,0)} P_0)$ is a tilting complex, then there are two isomorphisms of Ringel–Hall subalgebras:

(1) $\mathcal{H}(\mathcal{T}(M)) \cong \mathcal{H}(\mathcal{Y}(M))$,
(2) $\mathcal{H}(\mathcal{F}(M)) \cong \mathcal{H}(\mathcal{X}(M))$.

In particular, M is a tilting module or the torsion pair $(\mathcal{T}(M), \mathcal{F}(M))$ is split, and the above isomorphisms hold.

Proof. Let (M, P) be a support τ-tilting pair. Following [1, Theorem 3.2], we know that $\mathbf{P} = (P_1 \oplus P \xrightarrow{(f,0)} P_0)$ is a 2-term silting complex in $K^b(\mathrm{proj}\, A)$ such that $\mathrm{H}^0(\mathbf{P}) = M$, where $P_1 \xrightarrow{f} P_0$ is a minimal projective presentation of M. By [3, Proposition 2.4], $\mathcal{T}(\mathbf{P}) = \mathcal{T}(M)$ and so $(\mathcal{T}(\mathbf{P}), \mathcal{F}(\mathbf{P}))$=$(\mathcal{T}(M), \mathcal{F}(M))$. From [3, Lemma 2.7], we have a functorial isomorphism

$$\mathrm{Hom}_{D^b(A)}(\mathbf{P}, X) \cong \mathrm{Hom}_A(M, X),$$

for any $X \in$ mod A. Moreover, by [25, Corollary 2.2], the functor $\mathrm{Hom}_A(M, -) : \mathcal{T}(M) \to \mathcal{Y}(M)$ is an equivalence of categories. Thus, we have that $\mathcal{Y}(\mathbf{P}) = \mathcal{Y}(M)$ and hence $\mathcal{X}(\mathbf{P}) = \mathcal{X}(M)$. By Theorem 3.8, we have the isomorphisms.

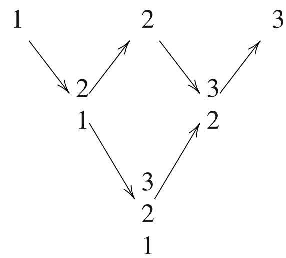
In particular, if $(\mathcal{T}(M), \mathcal{F}(M))$ is split, then $\mathbf{P}$ is separating and hence it is a tilting complex. If M is a tilting module, then $P = 0$ and $\mathbf{P}$ is just a tilting complex. It completes the proof. □

Finally, we provide an example to illustrate that Theorem 3.8 gives a proper generalization of obul's work.

Example 3.12. Let A be a path algebra given by the following quiver:

$$3 \longrightarrow 2 \longrightarrow 1\ .$$

The AR-quiver is given by



Let **P** be a 2-term complex given by the direct sums of the following complexes in $K^b(\text{proj } A)$:

$$\mathbf{P}_1 = 1 \to \begin{smallmatrix}2\\1\end{smallmatrix}, \qquad \mathbf{P}_2 = 1 \to \begin{smallmatrix}3\\2\\1\end{smallmatrix}, \qquad \mathbf{P}_3 = 1 \to 0.$$

It is easy to check that **P** is a silting complex. Let $\mathcal{T} = \mathcal{T}(\mathbf{P})$ and $\mathcal{F} = \mathcal{F}(\mathbf{P})$. Then one can get a splitting torsion pair $(\mathcal{T}, \mathcal{F})$ as follows:

$$\mathcal{T} = \text{add}\left\{2, \begin{smallmatrix}3\\2\end{smallmatrix}, 3\right\},$$
$$\mathcal{F} = \text{add}\left\{1, \begin{smallmatrix}2\\1\end{smallmatrix}, \begin{smallmatrix}3\\2\\1\end{smallmatrix}\right\}.$$

Thus, **P** is a separating silting complex and so is a tilting complex. Note that this torsion pair cannot be induced by any tilting module in mod A. Indeed, **P** is given by the support τ-tilting pair $\left(2 \oplus \begin{smallmatrix}3\\2\end{smallmatrix}, 1\right)$.

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