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Morphisms determined by objects under Galois G-covering theory

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ABSTRACT

The theory of morphisms being determined by objects was originally investigated by Auslander, and can be seen as the culmination part of Auslander-Reiten theory. This theory provides a more general frame for working with the Auslander-Reiten theory. In this paper, we will study the behavior of morphisms determined by objects under G-coverings in the sense of Asashiba, which is a generalization of Gabriel's Galois coverings. As an application, we reformulate Bautista and Liu's framework that normal G-coverings preserve sink maps and source maps. We also show that there is a G-covering between two relative stable categories, which unifies Asashiba, Hafezi, Vahed and Mahdavi's work on the stable categories. This is applied to the discussion on the existence of Serre functors by G-coverings of triangulated categories.

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1. Introduction

In 1978, Auslander introduced the concept of morphisms being determined by objects in his Philadelphia notes [5,6]. This theory is closely related to many aspects of representation theory of algebras, especially, Auslander-Reiten theory, which plays a central role in the representation theory of algebras. It gives a conceptual explanation for the conception of irreducible morphisms and the existence of left and right almost split morphisms from the categorical point of view. In fact, it provides an approach to construct and organize morphisms in additive categories, generalizing previous work of Auslander and Reiten on almost split sequences. Meanwhile, using the theory of morphisms being determined by objects, Auslander provided a method to construct or classify morphisms in a fixed category, that is, the so-called Auslander bijection. For this aspect, recently, many scholars have further developed and explained this theory, such as Ringel [30,31], Chen-Le [13], Zhao-Tan-Huang [36], etc.

Krause [24] showed that some of Auslander's results have an analogue for triangulated categories. For example, he showed that in a Hom-finite Krull-Schmidt triangulated category, the existence of the Serre functor is equivalent to that this category admits right determined morphisms. Jiao and Le [22] introduced the notions of having right (left) stably determined deflations (inflations) in exact categories. In that survey, they proved that in a Hom-finite Krull-Schmidt exact category, the existence of the Auslander-Reiten-Serre duality is equivalent to that this category admits right stably determined deflations. Recently, Zhao, Tan and Huang [35] unified these two versions of results into the extriangulated categories.

Covering theory, originated from topology, is widely used in algebraic topology [18]. Covering technique was introduced into representation theory of algebras and developed by Riedtmann [29], Bongartz-Gabriel [11], Gabriel [17], Dowbor-Lenzing-Skowroński [15], Waschbüsch [34], Martínez-Villa, de la Peña [27], Cibils-Eduardo [14], and so on. The classical Galois covering technique has been playing an important role in the representation theory of finite dimensional algebras. This theory reduces problems of the module category of an algebra A (whose structure is more complicated) to that of a category $\mathcal{C}$ with an action of a group G such that A is equivalent to the orbit category $\mathcal{C}/G$, which is easier to be understood. The category $\mathcal{C}$ is usually regarded as the “étale” of the algebra A.

The classical setting of covering technique requires the stringent conditions on categories, such as, local boundedness and freeness of group action. In fact, these assumptions are not easy to realize in general case. Thus it makes very inconvenient to apply the covering technique to usual additive categories such as the bounded homotopy category of finitely generated projective modules over a ring or even the module category. To overcome the difficulty, Asashiba [1] introduced the notion of G-(pre)coverings. Strengthening the notion of G-coverings, Bautista and Liu [9] defined the notion of Galois G-coverings for general linear categories. They showed that a Galois G-covering between Krull-Schmidt categories preserves irreducible morphisms and almost split sequences. Later, Bautista and Liu [10] showed that for a locally bounded linear category $\mathcal{A}$ with radical squared zero, the bounded derived category $D^b(\mathsf{mod}\mathcal{A})$ of finite dimensional left $\mathcal{A}$-modules admits a Galois G-covering. As an application of this result, they described the indecomposable objects of $D^b(\mathsf{mod}\mathcal{A})$ and obtained a complete description of the shapes of its Auslander-Reiten components. Recently, Asashiba, Hafezi and Vahed [2] constructed G-precoverings of bounded derived categories, singularity categories and Gorenstein defect categories. Then they obtained a Gorenstein version of Gabriel's theorem. Using this result, they investigated the number of summands in a decomposition of the middle term of almost split sequences over monomial algebras.

Inspired by these, one may naturally ask, as a generalization of almost split morphisms, how morphisms being determined by objects behave under G-coverings? Following the philosophy of Bautista and Liu [9], we expect to show that G-coverings preserve morphisms determined by objects.

The paper is organized as follows.

In Section 2, we will recall some terminologies and some preliminary results needed in this paper. We also define the notion of normal G-coverings, which is slightly different from Galois G-coverings in the sense of [9].

In Section 3, we prove that given a G-covering $F : \mathcal{A} \to \mathcal{B}$ between two Krull-Schmidt categories, F preserves morphisms determined by objects. On the other hand, under some suitable conditions, we prove that if $\mathcal{B}$ has right determined morphisms, then so does $\mathcal{A}$.

In Section 4, we apply the results in Section 3 to reformulate Bautista and Liu's framework. More precisely, for a normal G-covering F between two Krull-Schmidt categories, we prove that F preserves sink maps and source maps. It is remarkable that, compared with [9], our results do not require that the group action is locally bounded.

In Section 5, we give a G-covering between two relative stable categories, which unifies [2, Theorem 4.5] and [19, Proposition 2.6]. As an application, we provide a reduction technique for dealing with the existence of Serre functors in the stable categories of Gorenstein projective objects.

2. Preliminaries

Throughout this paper, all categories are skeletally small, and morphisms are composed from the right to the left. Let R be a commutative artin ring with radical $\mathfrak{r}$. An *R-linear category* is a category in which the morphism sets are R-modules such that the composition of morphisms is R-bilinear. All functors between R-linear categories are assumed to be R-linear. An R-linear category is called *additive* if it has finite direct sums. In the sequel, unless otherwise stated, a linear category refers to an R-linear category, an additive category refers to an additive R-linear category. Let $\mathcal{A}$ be a linear category. Denote by $\mathcal{A}_0$ the class of objects of $\mathcal{A}$.

Let k be a field. A k-linear category $\mathcal{A}$ is said to be *locally bounded* if it satisfies the following conditions:

(1) $\mathcal{A}$ is *basic* (i.e., distinct objects of $\mathcal{A}$ are not isomorphic);
(2) $\mathcal{A}$ is *semiperfect* (i.e., $\mathcal{A}(X,X)$ is a local algebra, for any $X \in \mathcal{A}_0$);
(3) For each $X \in \mathcal{A}_0$, $\sum_{Y\in\mathcal{A}_0} \dim_k \mathcal{A}(X,Y) < \infty$ and $\sum_{Y\in\mathcal{A}_0} \dim_k \mathcal{A}(Y,X) < \infty$.

A linear category is called *Hom-finite* if the morphism modules are of finite R-length. Moreover, a *Krull-Schmidt* category is an additive category in which every non-zero object is a finite direct sum of objects with local endomorphism algebras. An additive category has *split idempotents* if every idempotent endomorphism ϕ of an object X splits, that is, there exists a factorization $X \xrightarrow{u} Y \xrightarrow{v} X$ of ϕ with $uv = id_Y$ and $vu = \phi$.

It is well-known that $\mathcal{A}$ is a Krull-Schmidt category if and only if it has split idempotents and the endomorphism ring of every object is semi-perfect. In particular, in this case, an object $X \in \mathcal{A}$ is indecomposable if and only if $\mathrm{End}_{\mathcal{A}}(X)$ is local. For a Krull-Schmidt category $\mathcal{A}$, we denote by $\mathsf{ind}\mathcal{A}$ the full subcategory of representatives of the isomorphism classes of indecomposable objects in $\mathcal{A}$. For more details, one can refer for example to [25, Corollary 4.4] or [12, Theorem A.1].

Let $\mathcal{A}$ be a linear category. We define the category of left (resp. right) $\mathcal{A}$-modules, denoted by $\mathsf{Mod}\mathcal{A}$ (resp. $\mathsf{Mod}\mathcal{A}^{op}$), to be the functor category $\mathsf{Fun}(\mathcal{A}, \mathsf{Mod}R)$ (resp. $\mathsf{Fun}(\mathcal{A}^{op}, \mathsf{Mod}R)$) consisting of all covariant (resp. contravariant) functors. Moreover, $\mathsf{Mod}\mathcal{A}$ has enough projective objects, that is, for any $M \in \mathsf{Mod}\mathcal{A}$, there is an epimorphism $P \to M$ in $\mathsf{Mod}\mathcal{A}$, where P is a projective $\mathcal{A}$-module. For any $x \in \mathcal{A}_0$, the representable functor $P[x] = \mathcal{A}(x,-)$ is a projective $\mathcal{A}$-module in $\mathsf{Mod}\mathcal{A}$. We say that a left $\mathcal{A}$-module M is *finitely generated* if there is a finite family I of objects of $\mathcal{A}$ and an exact sequence

$$\oplus_{x\in I} P[x] \to M \to 0.$$

A locally bounded linear category $\mathcal{A}$ is said to be *Frobenius* (or *self-injective*) if every finitely generated projective $\mathcal{A}$-module is injective.

We say that a left $\mathcal{A}$-module M is *finitely presented* (or *coherent*) if there are finite families I, J of objects of $\mathcal{A}$ and an exact sequence

$$\oplus_{y\in J} P[y] \to \oplus_{x\in I} P[x] \to M \to 0.$$

We denote by $\mathsf{mod}\mathcal{A}$ the subcategory of $\mathsf{Mod}\mathcal{A}$ consisting of all finitely presented $\mathcal{A}$-modules.

Let $\mathsf{mod}R$ denote the category of finitely presented R-modules and fix an injective envelope $E = E(R/\mathfrak{r})$ over R. This provides the duality

$$D = \mathrm{Hom}_R(-, E) : \mathsf{mod}R \longrightarrow \mathsf{mod}R.$$

Recall that a Hom-finite Krull-Schmidt linear category $\mathcal{A}$ is called a *dualising R-variety* [7] if the assignment $F \to DF$ induces an equivalence

$$(\mathsf{mod}\mathcal{A})^{op} \xrightarrow{\sim} \mathsf{mod}(\mathcal{A}^{op}).$$

Let $\mathcal{A}$ be a linear category equipped with an action of a group G, that is, there exists a group homomorphism $\rho : G \to \mathrm{Aut}(\mathcal{A})$, where $\mathrm{Aut}(\mathcal{A})$ is the group of automorphisms of $\mathcal{A}$. Set $gX := \rho(g)(X)$, and $gf := \rho(g)(f)$ for any $g \in G$, $X, Y \in \mathcal{A}_0$ and $f \in \mathcal{A}(X, Y)$.

Definition 2.1 *([9]).* Let $\mathcal{A}$ be a linear category with G a group acting on $\mathcal{A}$. The G-action on $\mathcal{A}$ is called *admissible* if it satisfies the following conditions.

(1) The G-action is *free*, that is, $gX \ncong X$ for any indecomposable object X of $\mathcal{A}$ and any non-identity $g \in G$.
(2) The G-action is *locally bounded*, that is, for any indecomposable objects X and Y of $\mathcal{A}$, $\mathcal{A}(X, gY) = 0$ for all but finitely many $g \in G$.

Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between linear categories. Recall that for $g \in G$, a functorial (iso)morphism $\delta_g : F \circ g \to F$ consists of (iso)morphisms $\delta_{g,X} : (F \circ g)(X) \to F(X)$ for any $X \in \mathcal{A}_0$, which are natural in X.

Definition 2.2 *([1]).* Let $\mathcal{A}$, $\mathcal{B}$ be linear categories with G a group acting on $\mathcal{A}$. A functor $F : \mathcal{A} \to \mathcal{B}$ is called *G-stable* provided that there exist functorial isomorphisms $\delta_g : F \circ g \to F$ with all $g \in G$, such that the following diagram is commutative

$$\begin{array}{ccc} & & F(hX) = (F\circ h)(X) \\ & \overset{\delta_{g,hX}}{\nearrow} & \downarrow \delta_{h,X} \\ (F\circ gh)(X) = (F\circ g)(hX) & \xrightarrow{\delta_{gh,X}} & F(X), \end{array}$$

that is, $\delta_{h,X}\delta_{g,hX} = \delta_{gh,X}$ for any $g, h \in G$ and $X \in \mathcal{A}_0$. In this case, we call $\delta = (\delta_g)_{g\in G}$ a *G-stabilizer* for F.

Remark 2.3 *([9]).*

(1) By definition, $\delta^{-1}_{g,X} = \delta_{g^{-1},gX}$ for $g \in G$ and $X \in \mathcal{A}_0$; $\delta_e = id_F$, where e is the identity of G.
(2) A G-stable functor $F : \mathcal{A} \to \mathcal{B}$ is said to be *G-invariant* if the G-stabilizer δ satisfies $\delta_g = id_F$ for any $g \in G$.

Definition 2.4 *([1]).* Let $\mathcal{A}$, $\mathcal{B}$ be linear categories with G a group acting on $\mathcal{A}$. A functor $F : \mathcal{A} \to \mathcal{B}$ is called a *G-precovering* provided that F has a G-stabilizer δ such that, for any X, $Y \in \mathcal{A}_0$, the following two maps are isomorphisms:

$$F_{X,Y} : \bigoplus_{g\in G} \mathcal{A}(X, gY) \longrightarrow \mathcal{B}(F(X), F(Y)), \ \ (u_g)_{g\in G} \mapsto \sum_{g\in G} \delta_{g,Y} F(u_g)$$

$$F^{X,Y} : \bigoplus_{g\in G} \mathcal{A}(gX, Y) \longrightarrow \mathcal{B}(F(X), F(Y)), \ \ (v_g)_{g\in G} \mapsto \sum_{g\in G} F(v_g)\delta^{-1}_{g,X}.$$

Definition 2.5 *([1]).* Let $\mathcal{A}$, $\mathcal{B}$ be linear categories with G a group acting on $\mathcal{A}$. A G-precovering $F : \mathcal{A} \to \mathcal{B}$ is called a *G-covering* provided that F is dense, in the sense that for any $X' \in \mathcal{B}_0$, there exists an $X \in \mathcal{A}_0$ such that X' is isomorphic to $F(X)$ in $\mathcal{B}$.

Lemma 2.6 *([9]). Let $\mathcal{A}$, $\mathcal{B}$ be linear categories with G a group acting on $\mathcal{A}$ and let $F : \mathcal{A} \to \mathcal{B}$ be a G-precovering with a G-stabilizer δ.*

(1) *For any X, $Y \in \mathcal{A}_0$, we have the following decompositions*

$$\mathcal{B}(F(X), F(Y)) = \bigoplus_{g \in G} \delta_{g,Y} F(\mathcal{A}(X, gY)) = \bigoplus_{h \in G} F(\mathcal{A}(hX, Y))\delta^{-1}_{h,X}.$$

(2) *The functor F is faithful.*

Remark 2.7. By the direct sum decompositions in Lemma 2.6 (1), for any $u \in \mathcal{B}(F(X), F(Y))$, we can write $u = \sum_{g \in G} \delta_{g,Y} F(u_g) = \sum_{h \in G} F(v_h)\delta^{-1}_{h,X}$, where $u_g \in \mathcal{A}(X, gY)$ and $v_h \in \mathcal{A}(hX, Y)$ satisfy $u_g = 0$ and $v_h = 0$ for all but finitely many $g, h \in G$, respectively.

Lemma 2.8 *([9]). Let $\mathcal{A}$, $\mathcal{B}$ be linear categories with G a group acting on $\mathcal{A}$ and let $F : \mathcal{A} \to \mathcal{B}$ be a G-precovering. Consider a morphism $u : X \to Y$ in $\mathcal{A}$.*

(1) *If $v : X \to Z$ or $v : Z \to Y$ is a morphism in $\mathcal{A}$, then v factorizes through u if and only if $F(v)$ factorizes through $F(u)$.*
(2) *The morphism u is a section, a retraction, or an isomorphism if and only if $F(u)$ is a section, a retraction, or an isomorphism, respectively.*

Definition 2.9. Let $\mathcal{A}$, $\mathcal{B}$ be two linear categories with G a group acting freely on $\mathcal{A}$ and let $F : \mathcal{A} \to \mathcal{B}$ be a G-covering. Then F is said to be a *normal G-covering* provided that the following conditions are satisfied.

(1) For any $X \in \mathcal{A}_0$ with $\mathrm{End}_{\mathcal{A}}(X)$ is local, $\mathrm{End}_{\mathcal{B}}(F(X))$ is a local ring.
(2) Let X, $Y \in \mathcal{A}_0$ with $\mathrm{End}_{\mathcal{A}}(X)$ and $\mathrm{End}_{\mathcal{A}}(Y)$ local. If $F(X) \cong F(Y)$, then there exists some $g \in G$ such that $Y \cong gX$.

Definition 2.10 *([9, Definition 2.8]).* Let $\mathcal{A}$, $\mathcal{B}$ be two linear categories with G a group acting admissibly on $\mathcal{A}$ and let $F : \mathcal{A} \to \mathcal{B}$ be a G-covering. F is said to be a *Galois G-covering* provided that the following conditions are satisfied.

(1) If $X \in \mathcal{A}_0$ is indecomposable, then $F(X)$ is indecomposable.
(2) If X, $Y \in \mathcal{A}_0$ are indecomposable with $F(X) \cong F(Y)$, then there exists some $g \in G$ such that $Y \cong gX$.

Remark 2.11. If $\mathcal{A}$ is a Krull-Schmidt category and the G-action on $\mathcal{A}$ is admissible, then the notion of the normal G-covering coincides with that of the Galois G-covering.

Definition 2.12 *([1])*. Let $\mathcal{A}$ be a linear category with a G-action. The orbit category $\mathcal{A}/G$ of $\mathcal{A}$ by G is defined as follows.

(1) The class of objects of $\mathcal{A}/G$ is equal to that of $\mathcal{A}$.
(2) For each x, $y \in (\mathcal{A}/G)_0$, we set

$$\mathcal{A}/G(x,y) := (\Pi'(x,y))^G,$$

where

$$\Pi'(x,y) := \left\{ f = (f_{\beta,\alpha})_{(\alpha,\beta)} \in \prod_{(\alpha,\beta)\in G\times G} \mathcal{A}(\alpha x, \beta y) \middle| \; f \text{ is row finite and column finite} \right\}$$

and $(-)^G$ stands for the set of G-invariant elements, namely

$$(-)^G := \left\{ (f_{\beta,\alpha})_{(\alpha,\beta)} \in \Pi'(x,y) \middle| \; \forall\, \gamma \in \; G, \; f_{\gamma\beta,\gamma\alpha} = \gamma(f_{\beta,\alpha}) \right\}.$$

In the above, f is said to be row finite (resp. column finite) if for any $\alpha \in G$ the set $\{\beta \in G \mid f_{\alpha,\beta} \neq 0\}$ (resp. $\{\beta \in G \mid f_{\beta,\alpha} \neq 0\}$) is finite.
(3) For any composable morphisms $x \xrightarrow{f} y \xrightarrow{g} z$ in $\mathcal{A}/G$, we set

$$gf := \Big(\sum_{\gamma \in G} g_{\beta,\gamma} f_{\gamma,\alpha} \Big)_{(\alpha,\beta)\in G\times G} \in \mathcal{A}/G(x,z).$$

Remark 2.13. Let $\mathcal{A}$ be a locally bounded linear category. If the G-action on $\mathcal{A}$ is free, then the orbit category $\mathcal{A}/G$ defined above is equivalent to the classical orbit category $\mathcal{A}/_0G$ defined as follows:

(1) $(\mathcal{A}/_0G)_0 = \{Gx \mid x \in \mathcal{A}_0\}$;
(2) For any x, $y \in \mathcal{A}_0$, the morphism set $\mathcal{A}/_0G(Gx, Gy)$ is

$$\left\{ (f_{p,q}) \in \prod_{(p,q)\in Gx\times Gy} \mathcal{A}(p,q) \middle| \; f \text{ is row finite and column finite and } f_{aq,ap} = a(f_{q,p}), \; \forall\, a \in G \right\}.$$

Finally, assume that $\mathcal{A}$ is an additive category. The *radical* $\mathrm{rad}_{\mathcal{A}}(-,-)$ of $\mathcal{A}$ is the (two-sided) ideal of $\mathcal{A}$ defined by

$$\mathrm{rad}_{\mathcal{A}}(X,Y) := \{\, f \in \mathrm{Hom}_{\mathcal{A}}(X,Y) \mid id_X - gf \text{ is invertible for each } g : Y \to X\}$$

for any two objects X and Y in $\mathcal{A}$. A morphism $f : X \to Y$ is said to be radical if $f \in \mathrm{rad}_{\mathcal{A}}(X,Y)$. Furthermore, $\mathrm{rad}_{\mathcal{A}}(X,X) \subseteq \mathrm{End}_{\mathcal{A}}(X)$ coincides with the Jacobson radical $J(\mathrm{End}_{\mathcal{A}}(X))$ of the ring $\mathrm{End}_{\mathcal{A}}(X)$. It is well-known that if $X = X_1 \oplus X_2 \oplus \cdots \oplus X_n$ and $Y = Y_1 \oplus Y_2 \oplus \cdots \oplus Y_m$ are objects of $\mathcal{A}$, then a morphism

$$f = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{m1} & f_{m2} & \cdots & f_{mn} \end{bmatrix} : X \to Y,$$

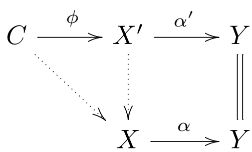
where each $f_{ji} \in \mathrm{Hom}_{\mathcal{A}}(X_i, Y_j)$, belongs to $\mathrm{rad}_{\mathcal{A}}(X,Y)$ if and only if each $f_{ji} \in \mathrm{rad}_{\mathcal{A}}(X_i, Y_j)$.

Let $\mathcal{A}$ be an additive category. For any two objects X, Y in $\mathcal{A}$ with $\mathrm{End}_{\mathcal{A}}(X)$ and $\mathrm{End}_{\mathcal{A}}(Y)$ local rings, $\mathrm{rad}_{\mathcal{A}}(X,Y)$ is the set of all non-isomorphisms from X to Y in $\mathcal{A}$. In particular, if $X \ncong Y$, then $\mathrm{rad}_{\mathcal{A}}(X,Y) = \mathrm{Hom}_{\mathcal{A}}(X,Y)$, see [3, Appendix A3]. For more details, one can see [8,23,25].

3. Morphisms determined by objects

Definition 3.1 *([5]).* Let $\mathcal{A}$ be an additive category. Fix an object C in $\mathcal{A}$. A morphism $\alpha : X \to Y$ in $\mathcal{A}$ is *right determined by the object* C if for every morphism $\alpha' : X' \to Y$ the following are equivalent:

(RD1) α' factors through α;
(RD2) for every $\phi : C \to X'$ the composite $\alpha'\phi$ factors through α.



A morphism is *left determined* by C if it is right determined by C when viewed as a morphism in the opposite category.

Remark 3.2. We denote by $\mathrm{Im}\,\mathrm{Hom}(C,\alpha)$ the image of $\mathrm{Hom}(C,\alpha) : \mathrm{Hom}(C,X) \to \mathrm{Hom}(C,Y)$. Then [RD2] means that $\mathrm{Im}\,\mathrm{Hom}(C,\alpha') \subseteq \mathrm{Im}\,\mathrm{Hom}(C,\alpha)$.

Example 3.3 *([24]).* Let $\mathcal{C}$ be a partially ordered set, viewed as a category, and fix a morphism $\alpha : x \to y$, which means that $x \le y$. If $x = y$, then α is right determined by every object of $\mathcal{C}$. If $x \ne y$, then α is right determined by an object $c \in \mathcal{C}$ if and only if there exists a unique minimal element in

$$\mathrm{C}_\alpha = \{z \in \mathcal{C} \mid z \nleq x, z \le y\}.$$

In that case $c = \inf \mathrm{C}_\alpha$. Thus in $(\mathbb{Z}, \leq)$ all morphisms are determined by objects, while in $(\mathbb{Q}, \leq)$ only identity morphisms are determined by some object.

Lemma 3.4. *Let $\mathcal{A}$, $\mathcal{B}$ be two additive categories with a G-action on $\mathcal{A}$ and $F : \mathcal{A} \to \mathcal{B}$ a G-covering. Assume that $\alpha : X \to Y$ is a morphism in $\mathcal{A}$. If α is right (resp. left) determined by an object C in $\mathcal{A}$, then $F(\alpha)$ is right (resp. left) determined by the object $F(C)$ in $\mathcal{B}$.*

Proof. Let $\alpha' : M \to F(Y)$ be any morphism in $\mathcal{B}$. Since F is dense, we may assume that $F(M') = M$. Suppose that

$$\mathrm{Im}\mathcal{B}(F(C), \alpha') \subseteq \mathrm{Im}\mathcal{B}(F(C), F(\alpha)). \tag{3.1}$$

That is, for any $\varphi : F(C) \to F(M')$, there exists a morphism $v : F(C) \to F(X)$ such that $F(\alpha)v = \alpha'\varphi$.

By Lemma 2.6, we may assume that $\alpha' = \Sigma_{i=1}^n F(\alpha_i')\delta_{g_i^{-1}, g_i M'}$, where $g_1, \cdots, g_n \in G$ are pairwise distinct and $\alpha_i' \in \mathcal{A}(g_i M', Y)$. Set $\tau = [\, \alpha_1' \ \alpha_2' \ \cdots \ \alpha_n' \,] : \oplus_{i=1}^n g_i M' \to Y$, and $G_0 = \{g_1, \cdots, g_n\} \subseteq G$.

Next, we claim that $\mathrm{Im}\mathcal{A}(C, \tau) \subseteq \mathrm{Im}\mathcal{A}(C, \alpha)$.

Note that $\mathcal{A}(C, \oplus_{i=1}^n g_i M') \cong \oplus_{i=1}^n \mathcal{A}(C, g_i M')$. Then for any $f \in \mathcal{A}(C, \oplus_{i=1}^n g_i M')$, we may write $f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$, where $f_i \in \mathcal{A}(C, g_i M')$. It suffices to find a morphism from C to X such that the following diagram commutes.

$$\begin{array}{ccccc} C & \xrightarrow{f} & \oplus_{i=1}^n g_i M' & \xrightarrow{\tau} & Y \\ & \searrow & & & \| \\ & & X & \xrightarrow{\alpha} & Y. \end{array}$$

Set $\phi = \Sigma_{i=1}^n \delta_{g_i, M'} F(f_i) \in \mathcal{B}(F(C), F(M'))$. Then by the assumption (3.1), there exists a morphism $v : F(C) \to F(X)$ such that $F(\alpha)v = \alpha'\phi$. By Lemma 2.6, we assume that $v = \Sigma_{h \in G} F(v_h)\delta_{h^{-1}, hC}$, where $v_h : hC \to X$ are zero morphisms for all but finitely many $h \in G$.

Observing that for any $1 \leq i, j \leq n$, and any $g_i, g_j \in G_0$,

$$\delta_{g_i^{-1}, g_i M'} \delta_{g_j, M'} = \delta_{g_i^{-1}, g_i M'} \delta_{g_j, g_i^{-1} g_i M'} = \delta_{g_j g_i^{-1}, g_i M'},$$

$$\delta_{g_j g_i^{-1}, g_i M'} F(f_j) = F(g_i g_j^{-1} f_j) \delta_{g_j g_i^{-1}, g_i g_j^{-1} C},$$

we have that

$$\begin{aligned}\alpha'\phi &= \sum_{i=1}^{n}\sum_{j=1}^{n} F(\alpha_i')\delta_{g_i^{-1},g_iM'}\delta_{g_j,M'}F(f_j)\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} F(\alpha_i')\delta_{g_jg_i^{-1},g_iM'}F(f_j)\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} F(\alpha_i')F(g_ig_j^{-1}f_j)\delta_{g_jg_i^{-1},g_ig_j^{-1}C}\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} F(\alpha_i'(g_ig_j^{-1}f_j))\delta_{g_jg_i^{-1},g_ig_j^{-1}C}.\end{aligned}$$

Set $\Omega_h = \{(i,j) \mid 1 \le i,j \le n,\ g_jg_i^{-1} = h\}$ for any $h \in G$. If Ω_h is an empty set for some $h \in G$, then we assume that $\sum_{\Omega_h} F(\alpha_i'(hf_j))\delta_{h^{-1},hC} = 0$. Then we have

$$\sum_{h\in G}\sum_{\Omega_h} F(\alpha_i'(hf_j))\delta_{h^{-1},hC} = \alpha'\phi = F(\alpha)v = \sum_{h\in G} F(\alpha v_h)\delta_{h^{-1},hC}.$$

Since $\mathcal{B}(F(C),F(Y)) = \bigoplus_{h\in G} F(\mathcal{A}(hC,Y))\delta_{h^{-1},hC}$, we deduce that for any $h \in G$,

$$F(\alpha v_h)\delta_{h^{-1},hC} = \sum_{\Omega_h} F(\alpha_i'(hf_j))\delta_{h^{-1},hC}.$$

In particular, we have that $F(\alpha v_e)\delta_{e,C} = \sum_{\Omega_e} F(\alpha_i'f_j)\delta_{e,C}$. Since $\delta_{e,C} = 1_C$ and F is faithful, we have that

$$\alpha v_e = \sum_{\Omega_e} \alpha_i'f_j.$$

Note that $\Omega_e = \{(i,j) \mid 1 \le i,j \le n, g_i = g_j\}$. In this case, we know that $i = j$ since $g_1, \cdots, g_n \in G$ are pairwise distinct. That is, $\alpha v_e = \sum_{i=1}^{n} \alpha_i'f_i = \tau f$. Hence, the claim holds.

Since α is right determined by the object C in $\mathcal{A}$, there exists a morphism $\mathbf{X} = [\, \gamma_1\ \gamma_2\ \cdots\ \gamma_n \,] : \oplus_{i=1}^{n} g_iM' \to X$ such that $\alpha\mathbf{X} = \tau$, where $\gamma_i : g_iM' \to X$. Thus, for any $1 \le i \le n$, $\alpha\gamma_i = \alpha_i'$. Therefore, $F(\alpha)\sum_{i=1}^{n} F(\gamma_i)\delta_{g_i^{-1},g_iM'} = \sum_{i=1}^{n} F(\alpha_i')\delta_{g_i^{-1},g_iM'} = \alpha'$. That is, α' factors through $F(\alpha)$. This completes the proof. □

Lemma 3.5. *Let $\mathcal{A}$, $\mathcal{B}$ be two additive categories with a G-action on $\mathcal{A}$ and $F : \mathcal{A} \to \mathcal{B}$ a G-covering. Assume that $C \in \mathcal{A}_0$ and $\alpha : X \to Y$ is a morphism in $\mathcal{A}$ such that each morphism in $\mathcal{A}(gC,Y)$ (resp. $\mathcal{A}(X,gC)$) factors through α, for any non-identity $g \in G$. If $F(\alpha)$ is right (resp. left) determined by the object $F(C)$ in $\mathcal{B}$, then α is right (resp. left) determined by the object C in $\mathcal{A}$.*

Proof. Let $\alpha' : X' \to Y$ be an arbitrary morphism in $\mathcal{A}$. Assume that

$$\mathrm{Im}\mathcal{A}(C, \alpha') \subseteq \mathrm{Im}\mathcal{A}(C, \alpha). \tag{3.2}$$

We claim that $\mathrm{Im}\mathcal{B}(F(C), F(\alpha')) \subseteq \mathrm{Im}\mathcal{B}(F(C), F(\alpha))$. It is enough to show that for any morphism $f : F(C) \to F(X')$, there is a morphism $v : F(C) \to F(X)$ such that $F(\alpha')f = F(\alpha)v$. Since F is a G-precovering, there is a family $\{f_g\}_{g\in G}$ such that $f = \sum_{g\in G} F(f_g)\delta_{g^{-1},gC}$ where $f_g \in \mathcal{A}(gC, X')$ for any $g \in G$. For the identity $e \in G$, since $\alpha' f_e : C \to Y$, by the assumption (3.2), there exists a morphism $v_e : C \to X$ such that $\alpha v_e = \alpha' f_e$. For each non-identity $g \in G$, since each $\theta \in \mathcal{A}(gC, Y)$ factors through α, $\alpha' f_g : gC \to Y$ factors through α. That is, there exists a morphism $v_g : gC \to X$ such that $\alpha v_g = \alpha' f_g$. Set $v = \sum_{g\in G} F(v_g)\delta_{g^{-1},gC} \in \mathcal{B}(F(C), F(X))$. Then $F(\alpha)v = F(\alpha')f$. Thus, the claim holds.

Since $F(\alpha)$ is right determined by the object $F(C)$, there is a morphism $u : F(X') \to F(X)$ such that $F(\alpha') = F(\alpha)u$. By Lemma 2.6, we may write $u = \sum_{i=1}^{n} F(u_i)\delta_{g_i^{-1},g_iX'}$ where $g_1, g_2, \cdots, g_n \in G$ are pairwise distinct, and $u_i \in \mathcal{A}(g_iX', X)$. Thus, we have

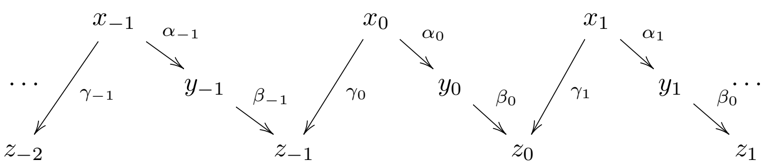
$$\sum_{i=1}^{n} F(\alpha u_i)\delta_{g_i^{-1},g_iX'} = F(\alpha)u = F(\alpha') = F(\alpha')\delta_{e,X'}.$$

By Lemma 2.6 (1), there exists some $1 \leq i_0 \leq n$ such that $g_{i_0} = e$, the identity of G. Since F is faithful, $\alpha u_{i_0} = \alpha'$, that is, α' factors through α. It completes the proof. □

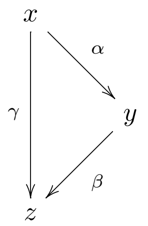
Proposition 3.6. *Let $\mathcal{A}$, $\mathcal{B}$ be two additive categories with a G-action on $\mathcal{A}$ and $F : \mathcal{A} \to \mathcal{B}$ a G-covering. Assume that $C \in \mathcal{A}_0$ and $\alpha : X \to Y$ is a morphism in $\mathcal{A}$. If α is right (resp. left) determined by the object C in $\mathcal{A}$, then $F(\alpha)$ is right (resp. left) determined by the object $F(C)$ in $\mathcal{B}$. The converse holds whenever each morphism in $F(\mathcal{A}(gC, Y))$ (resp. $F(\mathcal{A}(X, gC))$) factors through $F(\alpha)$, for any non-identity $g \in G$.*

Proof. By Lemma 2.8 (1), for each $\theta \in \mathcal{A}(gC, Y)$, θ factors through α if and only if $F(\theta)$ factors through $F(\alpha)$. The rest of the proof comes from Lemmas 3.4 and 3.5. □

Example 3.7. Let $k\widetilde{Q}$ and kQ be two path algebras over a field k, where the quiver $\widetilde{Q}$ is given by



and Q is given by



Then there is a natural $\mathbb{Z}$-action $\rho : \mathbb{Z} \to \mathrm{Aut}(k\widetilde{Q})$ on $k\widetilde{Q}$, given by $\rho(n)(v_i) = v_{i+n}$ and $\rho(n)(f_i) = f_{i+n}$ for any $v_i \in \{x_i, y_i, z_i \mid i \in \mathbb{Z}\}$ and $f_i \in \{\alpha_i, \beta_i, \gamma_i, \alpha_i\beta_i \mid i \in \mathbb{Z}\}$. We define a linear functor $\pi : k\widetilde{Q} \to kQ$ as follows. For each $v_i \in \{x_i, y_i, z_i \mid i \in \mathbb{Z}\}$ and $f_i \in \{\alpha_i, \beta_i, \gamma_i, \alpha_i\beta_i \mid i \in \mathbb{Z}\}$, $\pi(v_i) = v$ and $\pi(f_i) = f$ where $v \in \{x, y, z\}$ and $f \in \{\alpha, \beta, \gamma, \alpha\beta\}$. Regarding kQ and $k\widetilde{Q}$ as k-linear categories, it is easy to see that π is a $\mathbb{Z}$-covering. For any $v_i \in \{x_i, y_i, z_i \mid i \in \mathbb{Z}\}$ and non-zero $n \in \mathbb{Z}$, $k\widetilde{Q}(\rho(n)(v_i), v_i) = 0$ and $k\widetilde{Q}(v_i, \rho(n)(v_i)) = 0$. By definition, one can check that each morphism $x_i \to y_i$ is right determined by the object y_i in $k\widetilde{Q}$, for $i \in \mathbb{Z}$. Each morphism in $\pi(k\widetilde{Q}(\rho(n)(y_i), y_i)) = 0$ factors through $\pi(x_i \to y_i) = x \to y$, for any non-zero $n \in \mathbb{Z}$. Meanwhile, we can see that the morphism $x \to y$ is right determined by the object y in kQ.

Recall that a G-action on $\mathcal{A}$ is said to be *directed* [9] if for any indecomposable objects X and Y in $\mathcal{A}$, $\mathcal{A}(X, gY) = 0$ or $\mathcal{A}(gX, Y) = 0$ for all but at most one $g \in G$.

Corollary 3.8. *Let $\mathcal{A}$, $\mathcal{B}$ be two additive categories with a directed G-action on $\mathcal{A}$ and $F : \mathcal{A} \to \mathcal{B}$ a G-covering. Assume that $\alpha : X \to Y$ is a morphism in $\mathcal{A}$ with Y indecomposable* (*resp. X indecomposable*). *Then α is right* (*or left*) *determined by the object Y* (*resp. X*) *in $\mathcal{A}$ if and only if $F(\alpha)$ is right* (*resp. left*) *determined by the object $F(Y)$* (*resp. $F(X)$*) *in $\mathcal{B}$.*

Proof. Since G-action on $\mathcal{A}$ is directed, $\mathcal{A}(gY, Y) = 0$, for any non-identity $g \in G$. Thus, each morphism in $F(\mathcal{A}(gY, Y))$ factors through $F(\alpha)$, for any non-identity $g \in G$. It completes the proof. □

Next, we give a classical example to illustrate that there is a linear category with a directed G-action admitting a G-covering.

Example 3.9. Let kQ be the path algebra of the Dynkin quiver $1 \to 2 \to 3$ over an algebraically closed field k. Denote by $\mathcal{D} = \mathcal{D}^b(kQ)$ the bounded derived category of the finite dimensional (left) kQ-modules, [1] the shift functor in $\mathcal{D}$ and τ the Auslander-Reiten translation in $\mathcal{D}$. Then $g = \tau^{-1}[1]$ is an auto-isomorphism of $\mathcal{D}$. Set $G = \langle g \rangle$. It is easy to see that the G-action on $\mathcal{D}$ is admissible and directed. The orbit category $\mathcal{C} = \mathcal{D}/G$ is the so-called cluster category. Its objects are the G-orbits of the objects in $\mathcal{D}$. For each $X \in \mathcal{D}_0$, we denote by $\widetilde{X} = (g^i X)_{i \in \mathbb{Z}}$ its G-orbit. The following Fig. 1 is the Auslander-Reiten quiver $\Gamma_{\mathcal{D}}$ of $\mathcal{D}$, where the same color dots are in same G-orbits.

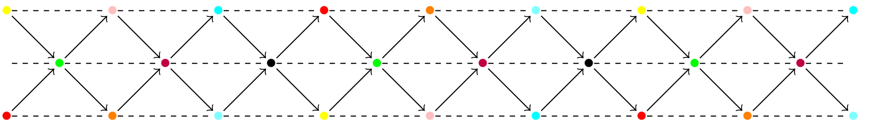


Fig. 1. The Auslander-Reiten quiver of $\mathcal{D}$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

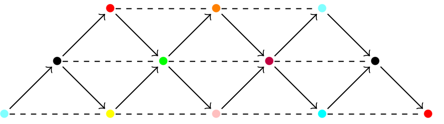


Fig. 2. The Auslander-Reiten quiver of $\mathcal{C}$.

We also have the Auslander-Reiten quiver $\Gamma_{\mathcal{C}}$ of $\mathcal{C}$, see Fig. 2.

It is well-known that the projection functor $\pi : \Gamma_{\mathcal{D}} \to \Gamma_{\mathcal{C}}$ is a G-invariant Galois G-covering, which sends each $X \in \mathcal{D}_0$ to its G-orbit.

Let k be an algebraically closed field. We denote $D = \mathrm{Hom}_k(-, k) : \mathsf{mod}k \to \mathsf{mod}k$ by the standard k-duality.

Proposition 3.10. *Let $\mathcal{A}$, $\mathcal{B}$ be two Hom-finite Krull-Schmidt k-categories with a G-action on $\mathcal{A}$ and $F : \mathcal{A} \to \mathcal{B}$ be a G-precovering. Assume that $C \in \mathcal{A}_0$ and $\alpha : X \to Y$ is a morphism in $\mathcal{A}$. If the morphism α satisfies the following conditions:*

(1) *there is an exact sequence*

$$\mathcal{B}(-, F(X)) \xrightarrow{(-,F(\alpha))} \mathcal{B}(-, F(Y)) \longrightarrow D\mathcal{B}(F(C), -); \tag{3.3}$$

(2) *for any non-identity $g \in G$, $\mathcal{A}(gC, Y) = 0$,*

then α is right determined by the object C in $\mathcal{A}$.

Proof. For any $Z' \in \mathcal{A}_0$, we have $F(Z') \in \mathcal{B}_0$. From the exact sequence (3.3), we obtain the following exact sequence

$$\mathcal{B}(F(Z'), F(X)) \xrightarrow{(F(Z'),F(\alpha))} \mathcal{B}(F(Z'), F(Y)) \longrightarrow D\mathcal{B}(F(C), F(Z')).$$

Note that $\oplus_{g\in G}\mathcal{A}(gC, Z') \cong \mathcal{B}(F(C), F(Z'))$ by Definition 2.4. Since $\mathcal{A}$ and $\mathcal{B}$ are Hom-finite, we have the isomorphism

$$\phi_{C,Z'} : \oplus_{g\in G} D(\mathcal{A}(gC, Z')) \xrightarrow{\sim} D\mathcal{B}(F(C), F(Z')).$$

By Definition 2.4 and Lemma 2.6, we have the following commutative diagram.

$$\begin{array}{ccccc}
\oplus_{g\in G}\mathcal{A}(Z',gX) & \xrightarrow{\oplus_{g\in G}(Z',g\alpha)} & \oplus_{g\in G}\mathcal{A}(Z',gY) & \longrightarrow & \oplus_{g\in G}D\mathcal{A}(gC,Z') \\
{\scriptstyle F_{Z',X}}\downarrow\cong & & {\scriptstyle F_{Z',Y}}\downarrow\cong & & {\scriptstyle \phi_{C,Z'}}\downarrow\cong \\
\mathcal{B}(F(Z'),F(X)) & \xrightarrow{(F(Z'),F(\alpha))} & \mathcal{B}(F(Z'),F(Y)) & \longrightarrow & D\mathcal{B}(F(C),F(Z')).
\end{array}$$

Then the first row is exact. Hence, we have the following exact sequence

$$\oplus_{g\in G}\mathcal{A}(-,gX) \xrightarrow{\oplus_{g\in G}(-,g\alpha)} \oplus_{g\in G}\mathcal{A}(-,gY) \longrightarrow \oplus_{g\in G}D\mathcal{A}(gC,-).$$

It implies that Coker $\oplus_{g\in G}(-,g\alpha) \cong \oplus_{g\in G}\text{Coker}(-,g\alpha)$ can be embedded into $\oplus_{g\in G}D\mathcal{A}(gC,-)$. Thus, Coker$(-,\alpha)$ can be embedded into $\oplus_{g\in G}D\mathcal{A}(gC,-)$. Let η : Coker$(-,\alpha) \to \oplus_{g\in G}D\mathcal{A}(gC,-)$ be the inclusion. Then one can write η as a column matrix $[\eta_g]_{g\in G}$, where η_g : Coker$(-,\alpha) \to D\mathcal{A}(gC,-)$. For any non-identity $h\in G$, applying the functor $\mathsf{Fun}(-,D\mathcal{A}(hC,-))$ to the following exact sequence

$$\mathcal{A}(-,X) \xrightarrow{(-,\alpha)} \mathcal{A}(-,Y) \longrightarrow \text{Coker}(-,\alpha) \longrightarrow 0,$$

we have an exact sequence

$$0 \longrightarrow \mathsf{Fun}(\text{Coker}(-,\alpha), D\mathcal{A}(hC,-)) \longrightarrow \mathsf{Fun}(\mathcal{A}(-,Y), D\mathcal{A}(hC,-)).$$

By Yoneda Lemma, we know that $\mathsf{Fun}(\mathcal{A}(-,Y),D\mathcal{A}(hC,-)) \cong D\mathcal{A}(hC,Y) = 0$, by the assumption (2). Then we obtain that $\mathsf{Fun}(\text{Coker}(-,\alpha),D\mathcal{A}(hC,-)) = 0$ and so η_h : Coker$(-,\alpha) \to D\mathcal{A}(hC,-)$ is zero. Hence, η_e : Coker$(-,\alpha) \to D\mathcal{A}(C,-)$ is injective. Therefore, we have the following exact sequence

$$\mathcal{A}(-,X) \xrightarrow{(-,\alpha)} \mathcal{A}(-,Y) \longrightarrow D\mathcal{A}(C,-).$$

Form [24, Proposition 3.7], we have that α is right determined by C. □

Corollary 3.11. *Let $\mathcal{A}$, $\mathcal{B}$ be two Hom-finite Krull-Schmidt k-categories with a directed G-action on $\mathcal{A}$ and $F:\mathcal{A}\to\mathcal{B}$ a G-precovering. Assume that $\alpha: X\to Y$ is a morphism in $\mathcal{A}$ with Y indecomposable. If there is an exact sequence*

$$\mathcal{B}(-,F(X)) \xrightarrow{(-,F(\alpha))} \mathcal{B}(-,F(Y)) \longrightarrow D\mathcal{B}(F(Y^m),-),$$

where m is an integer, then α is right determined by the object Y in $\mathcal{A}$.

Proof. Since the G-action on $\mathcal{A}$ is directed and Y is indecomposable, we have $\mathcal{A}(gY^m,Y) \cong \mathcal{A}(gY,Y)^m = 0$ for any non-identity $g\in G$. By Proposition 3.10, we have that α is right determined by the object Y^m in $\mathcal{A}$.

Next, we claim that α is right determined by the object Y.

In fact, for any $\alpha' : X' \to Y$, we assume that $\mathrm{Im}\mathcal{A}(Y, \alpha') \subseteq \mathrm{Im}\mathcal{A}(Y, \alpha)$. Then we have $\mathrm{Im}\mathcal{A}(Y^m, \alpha') \subseteq \mathrm{Im}\mathcal{A}(Y^m, \alpha)$. Since α is right determined by the object Y^m in $\mathcal{A}$, we know that α' factors through α. Thus, α is right determined by the object Y. □

Proposition 3.12. *Let $\mathcal{A}$, $\mathcal{B}$ be two Hom-finite Krull-Schmidt k-categories with G a group acting freely and directly on $\mathcal{A}$, and $F : \mathcal{A} \to \mathcal{B}$ a G-covering. Assume that $\alpha : X \to Y$ is a morphism in $\mathcal{A}$ with Y indecomposable. If the following conditions hold:*

(1) *$F(\alpha)$ is right determined by $F(Y)$;*
(2) *the action of G on $\mathsf{ind}\mathcal{A}$ has only finitely many G-orbits,*

then α is right determined by Y in $\mathcal{A}$.

Proof. Note that F is a G-covering functor between categories $\mathcal{A}$ and $\mathcal{B}$. Then, it induces the G-covering functor between their indecomposable objects, denoted by $\widetilde{F} : \mathsf{ind}\mathcal{A} \to \mathsf{ind}\mathcal{B}$. Since the G-action is free, by [1, Theorem 2.9], there is an equivalence $\mathsf{ind}\mathcal{B} \cong (\mathsf{ind}\mathcal{A})/G$, where $(\mathsf{ind}\mathcal{A})/G$ is the orbit category of $\mathcal{A}$. Since the action of G on $\mathsf{ind}\mathcal{A}$ has only finitely many G-orbits, we know that $\mathsf{ind}\mathcal{B}$ has finitely many objects. Let $\{X_1, X_2, \cdots, X_n\}$ be all pairwise nonisomorphic objects in $\mathsf{ind}\mathcal{B}$. Since $\mathcal{B}$ is a Krull-Schmidt category, we know that $\mathcal{B} = \mathrm{add}M$, where $M = \oplus_{i=1}^n X_i$. Thus, $\mathsf{mod}\mathcal{B}^{op}$ is equivalent to the category of finitely generated modules over the finite dimensional k-algebra $\mathrm{End}_{\mathcal{B}}(M)$. Thus, $\mathcal{B}$ is a dualising k-variety.

Consider the exact sequence

$$\mathcal{B}(-, F(X)) \xrightarrow{(-,F(\alpha))} \mathcal{B}(-, F(Y)) \longrightarrow \mathrm{Coker}(-, F(\alpha)) \longrightarrow 0.$$

Assume that $\mathrm{Coker}(-, F(\alpha)) \neq 0$. Let $E(\mathrm{Coker}(-, F(\alpha)))$ be the injective envelope of $\mathrm{Coker}(-, F(\alpha))$. Then, $E(\mathrm{Coker}(-, F(\alpha))) \cong D\mathcal{B}(Z, -)$, where Z is an object in $\mathcal{B}$. Hence, we have the following exact sequence

$$\mathcal{B}(-, F(X)) \xrightarrow{(-,F(\alpha))} \mathcal{B}(-, F(Y)) \longrightarrow D\mathcal{B}(Z, -).$$

By [24, Proposition 3.7], $F(\alpha)$ is right determined by Z. By the assumption (1) and [24, Proposition 3.13], $\mathrm{add}Z \subseteq \mathrm{add}F(Y)$. Then there is an object Z' such that $Z \oplus Z' = F(Y^m)$ for some integer m.

Then, we have the following exact sequence

$$\mathcal{B}(-, F(X)) \xrightarrow{(-,F(\alpha))} \mathcal{B}(-, F(Y)) \longrightarrow D\mathcal{B}(F(Y^m), -). \tag{3.4}$$

By Corollary 3.11, α is right determined by the object Y.

If $\mathrm{Coker}(-, F(\alpha)) = 0$, one can get the exact sequence (3.4) directly. By similar argument, we obtain the results. □

Let A be a finite dimensional k-algebra. In the classical theory of morphisms determined by objects, given a morphism $f : X \to Y$ which is right determined by C in $\mathsf{mod}A$, the right $\mathrm{End}_A(C)$-submodule $\mathrm{Im}\mathrm{Hom}_A(C, f)$ of $\mathrm{Hom}_A(C, Y)$ plays a crucial role. Auslander [5] made a great effort to give a positive answer to the question that for any modules C, Y and any right $\mathrm{End}_A(C)$-submodule H of $\mathrm{Hom}_A(C, Y)$, is there a morphism $f : X \to Y$ determined by C such that $\mathrm{Im}\mathrm{Hom}_A(C, f) = H$? Krause [24] provided a similar result for triangulated categories.

Now, we end this section by giving the following result.

Proposition 3.13. *Let $\mathcal{A}$, $\mathcal{B}$ be two Hom-finite Krull-Schmidt categories with G a group acting freely on $\mathcal{A}$ and $F : \mathcal{A} \to \mathcal{B}$ a G-covering. Assume that Y, C are two objects in $\mathcal{A}$ such that $\mathcal{A}(gC, Y) = 0$ for each non-identity $g \in G$. If the action of G on* $\mathsf{ind}\mathcal{A}$ *has only finitely many G-orbits, then for any right* $\mathrm{End}_{\mathcal{A}}(C)$*-submodule $H \subseteq \mathcal{A}(C, Y)$, there is a morphism $\alpha : X \to Y$ such that* $\mathrm{Im}\mathcal{A}(C, \alpha) = H$.

Proof. Since F is a G-precovering, F is faithful. Then for objects Y, C in $\mathcal{A}$, F induces the monomorphism $F_e = F_{C,Y} \circ inc_e : \mathcal{A}(C, Y) \to \mathcal{B}(F(C), F(Y))$ given by $F_e(u) = F(u)$ for any $u : C \to Y$, where $inc_e : \mathcal{A}(C, Y) \to \oplus_{g\in G}\mathcal{A}(C, gY)$ is the canonical embedding and $F_{C,Y} : \oplus_{g\in G}\mathcal{A}(C, gY) \to \mathcal{B}(F(C), F(Y))$ is the isomorphism defined in Definition 2.4. For any right $\mathrm{End}_{\mathcal{A}}(C)$-submodule $H \subseteq \mathcal{A}(C, Y)$, there is a monomorphism $F_e \circ \iota : H \to \mathcal{B}(F(C), F(Y))$, where $\iota : H \to \mathcal{A}(C, Y)$ is the inclusion. Note that the composition map $\mathcal{A}(C, Y) \times \mathcal{A}(gC, C) \to \mathcal{A}(gC, Y)$ is zero for every non-identity g since $\mathcal{A}(gC, Y) = 0$ for any every non-identity $g \in G$. Hence, H can be viewed as a right $\mathrm{End}_{\mathcal{B}}(F(C))$-submodule H of $\mathcal{B}(F(C), F(Y))$. Note that $\mathsf{mod}\mathrm{End}_{\mathcal{B}}(F(C))^{op}$ is an abelian category with enough projective and injective modules. Then there is an injective embedding $\mathcal{B}(F(C), F(Y))/H \hookrightarrow (D\mathrm{End}_{\mathcal{B}}(F(C)))^m$ in $\mathsf{mod}\mathrm{End}_{\mathcal{B}}(F(C))^{op}$. Let $\Gamma_C = \mathrm{End}_{\mathcal{B}}(F(C))^{op}$. By [24, 3.2], there is a morphism in the category $\mathsf{mod}\mathcal{B}^{op}$ consisting of coherent functors

$$\eta : \mathcal{B}(-, F(Y)) \longrightarrow \mathrm{Hom}_{\Gamma_C}(\mathcal{B}(F(C), -), D\mathrm{End}_{\mathcal{B}}(F(C)))^m.$$

By [24, Lemma 3.5], we know that

$$\mathrm{Hom}_{\Gamma_C}(\mathcal{B}(F(C), -), D\mathrm{End}_{\mathcal{B}}(F(C))) \cong D\mathcal{B}(F(C), -).$$

Thus, we can write $\eta : \mathcal{B}(-, F(Y)) \to D\mathcal{B}(F(C^m), -)$ since F is additive. In particular, $\ker \eta_{F(C)} = H$.

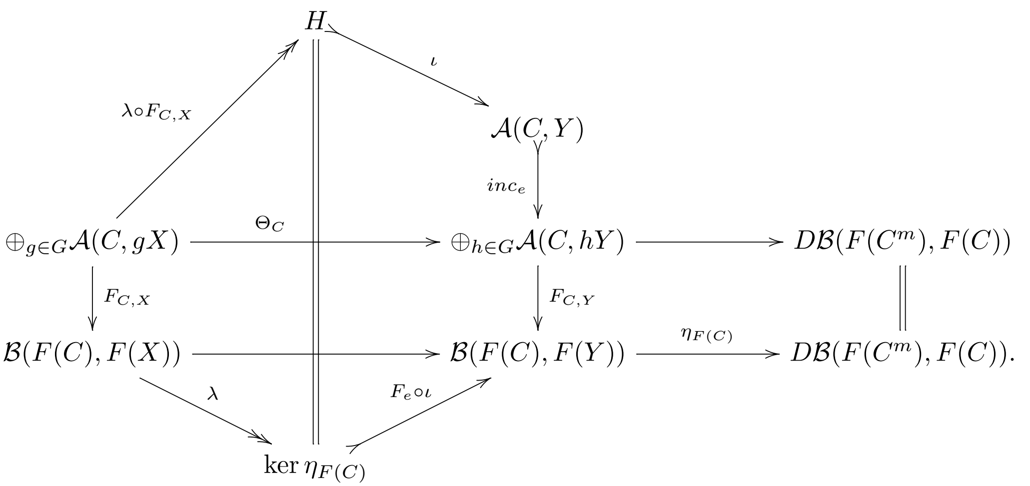
Note that the action of G on $\mathsf{ind}\mathcal{A}$ has only finitely many G-orbits. Then, by Proposition 3.12, $\mathsf{mod}\mathcal{B}^{op}$ is an abelian category with enough projective objects. Thus, $\ker \eta \in \mathsf{mod}\mathcal{B}^{op}$. Take a projective cover $\mathcal{B}(-, X') \to \ker \eta$ for some $X' \in \mathcal{B}_0$. Since F is dense, we assume that $X' = F(X)$ for some $X \in \mathcal{A}_0$. Then there is an exact sequence

$$\mathcal{B}(-, F(X)) \longrightarrow \mathcal{B}(-, F(Y)) \xrightarrow{\eta} D\mathcal{B}(F(C^m), -).$$

By Definition 2.4, we have the following commutative diagram

$$\begin{array}{ccccc}
\oplus_{g\in G}\mathcal{A}(-,gX) & \xrightarrow{\Theta} & \oplus_{h\in G}\mathcal{A}(-,hY) & \longrightarrow & D\mathcal{B}(F(C^m),-) \\
\downarrow{\scriptstyle F_{-,X}} & & \downarrow{\scriptstyle F_{-,Y}} & & \| \\
\mathcal{B}(-,F(X)) & \longrightarrow & \mathcal{B}(-,F(Y)) & \longrightarrow & D\mathcal{B}(F(C^m),-).
\end{array}$$

We may write $\Theta = [\widehat{\theta^{h,g}}]_{h\times g\in G\times G}$. By Yoneda Lemma, for any $g,h \in G$, there is a morphism $\theta^{h,g} : gX \to hY$ such that $\widehat{\theta^{h,g}} = \mathcal{A}(-,\theta^{h,g}) : \mathcal{A}(-,gX) \to \mathcal{A}(-,hY)$. In particular, we have the following commutative diagram.



From the commutative diagram, we have $\mathrm{Im}\Theta_C = H \subseteq \mathcal{A}(C,Y)$. It implies that $\mathrm{Im}\Theta_C = \mathrm{Im}[\widehat{\theta_C^{e,g}}]_{g\in G}$. Since $\mathcal{A}$, $\mathcal{B}$ are Hom-finite, there exists a finite subset G_0 of G such that $\oplus_{g\in G}\mathcal{A}(C,gX) = \oplus_{g\in G_0}\mathcal{A}(C,gX)$ and $\mathcal{A}(C,g'X) = 0$ for $g' \in G\backslash G_0$. Thus, $\mathrm{Im}\Theta_C = \mathrm{Im}[\widehat{\theta_C^{e,g}}]_{g\in G} = \mathrm{Im}[\widehat{\theta_C^{e,g}}]_{g\in G_0}$. Note that $[\widehat{\theta_C^{e,g}}]_{g\in G_0} = \mathcal{A}(C,\oplus_{g\in G_0}\theta_C^{e,g})$. Set $\alpha = \oplus_{g\in G_0}\theta_C^{e,g} : \oplus_{g\in G_0}gX \to Y$. Then we have that $\mathrm{Im}\mathcal{A}(C,\alpha) = H$. This completes the proof. □

Remark 3.14. Unfortunately, we could not be sure whether or not the defined morphism α is determined by some object.

4. Applying into Auslander-Reiten theory

Let $\mathcal{T}$ be a Hom-finite Krull-Schmidt triangulated category over an algebraically closed field k. Recall from [28] that a *right Serre functor* on $\mathcal{T}$ is an additive functor $S : \mathcal{T} \to \mathcal{T}$ together with a natural isomorphism $D\mathcal{T}(X,-) \cong \mathcal{T}(-,SX)$ for any object X in $\mathcal{T}$, where D is the standard k-duality. A right Serre functor is said to be *Serre* if it is an equivalence. In this section, our aim is to show that for a G-covering functor $F : \mathcal{A} \to \mathcal{B}$ between two Hom-finite Krull-Schmidt triangulated categories, $\mathcal{B}$ has Serre functor whenever $\mathcal{A}$ has Serre functor.

Recall from [20, 4.5] that $\mathcal{T}$ has sink morphisms (resp. has source morphisms) if, for any indecomposable object in $\mathcal{T}$, there exists a sink morphism (resp. a source morphism) in $\mathcal{T}$. From [28, Theorem I.2.4] and [20], we know that $\mathcal{T}$ has Serre functor if and only if $\mathcal{T}$ has sink morphisms and source morphisms.

Therefore, the problem turns into showing whether the functor F preserves sink morphisms and source morphisms. In fact, it should be attributed to Bautista and Liu's work in [9]. However, in this section, we will apply the theory of morphisms determined by objects obtained in Section 3 to show that a G-covering functor preserves sink morphisms and source morphisms, which provides a quite different method from [9].

Let $\mathcal{A}$ be an additive category. Recall that a morphism $f : X \to Y$ in $\mathcal{A}$ is said to be a *right almost split* morphism if it satisfies the following conditions:

(1) f is not a retraction;
(2) every morphism $f' : X' \to Y$, which is not a retraction, factors through f.

Moreover, if f is both right almost split and *right minimal* (i.e. each $h \in \mathrm{End}_{\mathcal{A}}(Y)$ satisfying $fh = f$ is an automorphism), then we call it a *sink* morphism. Dually, one can define the notions of *left almost split*, *left minimal* and *source* morphisms.

Proposition 4.1 *([5, Sect. II.2]). A morphism $f : X \to Y$ in an additive category $\mathcal{A}$ is right almost split if and only if f satisfies the following conditions:*

(1) $\mathrm{End}_{\mathcal{A}}(Y)$ *is a local ring;*
(2) f *is right determined by the object Y in $\mathcal{A}$;*
(3) $\mathrm{Im}\mathcal{A}(Y, f) = \mathrm{rad}_{\mathcal{A}}(Y, Y)$.

Proposition 4.2 *([26, Corollary 1.4]). Let $\phi : M \to N$ be a morphism in an additive category $\mathcal{A}$ and suppose that idempotents in $\mathrm{End}_{\mathcal{A}}(M)$ split. If M is a finite direct sum of indecomposable objects with local endomorphism rings, then there exists a decomposition*

$$\phi = [\,\phi' \ \phi''\,] : M = M' \oplus M'' \longrightarrow N$$

such that ϕ' is right minimal and $\phi'' = 0$.

Lemma 4.3. *Let $\mathcal{A}$, $\mathcal{B}$ be two Krull-Schmidt categories with G a group acting freely on $\mathcal{A}$ and $F : \mathcal{A} \to \mathcal{B}$ a G-covering. Assume that $u : X \to Y$ is a morphism in $\mathcal{A}$. If $F(u)$ is right (resp. left) almost split in $\mathcal{B}$, then u is right (resp. left) almost split in $\mathcal{A}$.*

Proof. Assume that $F(u)$ is right almost split in $\mathcal{B}$. Because F is a G-precovering, there is a monomorphism $F_e : \mathrm{End}_{\mathcal{A}}(Y) \to \mathrm{End}_{\mathcal{B}}(F(Y))$ given by $F_e(f) = F(f)$ for any

$f \in \mathrm{End}_{\mathcal{A}}(Y)$. Since $\mathrm{End}_{\mathcal{B}}(F(Y))$ is a local ring and $\mathcal{A}$ is a Krull-Schmidt category, we have that $\mathrm{End}_{\mathcal{A}}(Y)$ is local from the monomorphism F_e. If g is a non-identity element of G, then $gY \ncong Y$ since the G-action is free. Thus, for any $w \in \mathrm{Hom}_{\mathcal{A}}(gY, Y)$, $w \in \mathrm{rad}_{\mathcal{A}}(gY, Y)$, since $\mathrm{End}_{\mathcal{A}}(gY)$ is local, we have that w is a non-retraction by [32, Proposition 3.11]. By Lemma 2.8 (2), $F(w)$ is a non-retraction. Since $F(u)$ is right almost split, $F(w)$ factors through $F(u)$. That is, each morphism in $F(\mathcal{A}(gY, Y))$ factors through $F(u)$. By Proposition 3.6, u is right determined by the object Y in $\mathcal{A}$.

Since $F(u)$ is a non-retraction, by Lemma 2.8 (2), u is a non-retraction. By [32, Proposition 3.11], $\mathrm{Im}\mathcal{A}(Y, u) \subseteq \mathrm{rad}_{\mathcal{A}}(Y, Y)$. For any $p \in \mathrm{rad}_{\mathcal{A}}(Y, Y)$, p is a non-isomorphism, and so is $F(p)$. Then $F(p) \in \mathrm{rad}_{\mathcal{B}}(F(Y), F(Y))$ since $\mathrm{End}_{\mathcal{B}}(F(Y))$ is a local ring. Since $\mathrm{Im}\mathcal{B}(F(Y), F(u)) = \mathrm{rad}_{\mathcal{B}}(F(Y), F(Y))$, there exists a morphism $p' : F(Y) \to F(X)$ such that $F(p) = F(u)p'$. By Lemma 2.6 (1), we may assume that $p' = \sum_{i=1}^{n} F(p'_i)\delta^{-1}_{g_i,Y}$, where $g_1, g_2, \cdots, g_n \in G$ are pairwise distinct and $p'_i \in \mathcal{A}(g_iY, X)$. Then, we have

$$F(p) = \sum_{i=1}^{n} F(up'_i)\delta^{-1}_{g_i,Y}.$$

Thus, there exists some $1 \leq i_0 \leq n$ such that $g_{i_0} = e$, the identity element of G, and $F(p) = F(up'_{i_0})$. Since F is faithful, $p = up'_{i_0}$. That is, p factors through u and so $\mathrm{rad}_{\mathcal{A}}(Y, Y) \subseteq \mathrm{Im}\mathcal{A}(Y, u)$. Hence, $\mathrm{rad}_{\mathcal{A}}(Y, Y) = \mathrm{Im}\mathcal{A}(Y, u)$. It completes the proof. □

Lemma 4.4. *Let $\mathcal{A}$, $\mathcal{B}$ be two Krull-Schmidt categories with G a group acting freely on $\mathcal{A}$ and $F : \mathcal{A} \to \mathcal{B}$ a normal G-covering. Assume that $u : X \to Y$ is a morphism in $\mathcal{A}$. If u is right (resp. left) almost split in $\mathcal{A}$, then $F(u)$ is right (resp. left) almost split in $\mathcal{B}$.*

Proof. Assume that u is right almost split. Then u is right determined by the object Y in $\mathcal{A}$ and so $F(u)$ is right determined by the object $F(Y)$ in $\mathcal{B}$ by Proposition 3.6. Since F is a normal G-covering and $\mathrm{End}_{\mathcal{A}}(Y)$ is local, we know that $\mathrm{End}_{\mathcal{A}}(F(Y))$ is local.

Next, we shall prove that $\mathrm{Im}\mathcal{B}(F(Y), F(u)) = \mathrm{rad}_{\mathcal{B}}(F(Y), F(Y))$.

First of all, we check that $\mathrm{Im}\mathcal{B}(F(Y), F(u)) \subseteq \mathrm{rad}_{\mathcal{B}}(F(Y), F(Y))$. For any morphism $f : F(Y) \to F(X)$, we may write $f = \sum_{g\in G} F(f_g)\delta^{-1}_{g,Y}$, where $f_g \in \mathcal{A}(gY, X)$ for any $g \in G$. Then $F(u)f = \sum_{g\in G} F(uf_g)\delta^{-1}_{g,Y} \in \mathrm{End}_{\mathcal{B}}(F(Y))$ and $uf_g \in \mathcal{A}(gY, Y)$ for any $g \in G$. For any non-identity $g \in G$, since the G-action on $\mathcal{A}$ is free, $gY \ncong Y$. Then, we know that $uf_g \in \mathrm{rad}_{\mathcal{A}}(gY, Y)$ since $\mathrm{End}_{\mathcal{A}}(Y)$ is local. That is, uf_g is non-invertible and so is $F(uf_g)$ by Lemma 2.8 (2). Thus, $F(uf_g) \in \mathrm{rad}_{\mathcal{B}}(F(gY), F(Y))$ since $\mathrm{End}_{\mathcal{B}}(F(Y))$ and $\mathrm{End}_{\mathcal{B}}(F(gY))$ are local. Then $F(uf_g)\delta^{-1}_{g,Y} \in \mathrm{rad}_{\mathcal{B}}(F(Y), F(Y))$. For the identity $e \in G$, $uf_e \in \mathrm{Im}\mathcal{A}(Y, u)$. Since u is right almost split, $\mathrm{Im}\mathcal{A}(Y, u) = \mathrm{rad}_{\mathcal{A}}(Y, Y)$. Then $uf_e \in \mathrm{rad}_{\mathcal{A}}(Y, Y)$, and thus uf_e is non-invertible since $\mathrm{End}_{\mathcal{A}}(Y)$ is local. It follows that $F(uf_e)$ is non-invertible by Lemma 2.8 (2). Then $F(uf_e) \in \mathrm{rad}_{\mathcal{B}}(F(Y), F(Y))$ since $\mathrm{End}_{\mathcal{A}}(F(Y))$ is local. Hence, $F(u)f \in \mathrm{rad}_{\mathcal{B}}(F(Y), F(Y))$.

Secondly, we prove that $\mathrm{rad}_{\mathcal{B}}(F(Y), F(Y)) \subseteq \mathrm{Im}\mathcal{B}(F(Y), F(u))$.

For any $f \in \mathrm{rad}_{\mathcal{B}}(F(Y), F(Y))$, we may write that $f = \sum_{g\in G} F(f_g)\delta^{-1}_{g,Y}$, where $f_g \in \mathcal{A}(gY, Y)$ for any $g \in G$. For any non-identity $g \in G$, since the G-action on $\mathcal{A}$ is free, we have $gY \not\cong Y$. Then $f_g \in \mathrm{rad}_{\mathcal{A}}(gY, Y)$ by the fact that $\mathrm{End}_{\mathcal{A}}(gY)$ and $\mathrm{End}_{\mathcal{A}}(Y)$ are local. Thus, f_g is not a retraction. Note that u is right almost split. This gives rise to $f_g = uh_g$ for some morphism $h_g : gY \to X$. For the identity $e \in G$, we claim that f_e is non-invertible. Indeed, suppose that f_e is invertible, and so is $F(f_e)$ by Lemma 2.8 (2). Then

$$f = F(f_e) + \sum_{\substack{g\in G\\ g\neq e}} F(f_g)\delta_{g,Y}^{-1}.$$

Note that $f_g \in \mathrm{rad}_{\mathcal{A}}(gY, Y)$ for any non-identity $g \in G$. Thus, f_g is non-invertible, and so is $F(f_g)$ by Lemma 2.8 (2). Since $\mathrm{End}_{\mathcal{B}}(F(gY))$ and $\mathrm{End}_{\mathcal{B}}(F(Y))$ are local, $F(f_g) \in \mathrm{rad}_{\mathcal{B}}(F(gY), F(Y))$ and so $\sum_{\substack{g\in G\\ g\neq e}} F(f_g)\delta_{g,Y}^{-1} \in \mathrm{rad}_{\mathcal{B}}(F(Y), F(Y))$. It implies that $F(f_e) = (f - \sum_{\substack{g\in G\\ g\neq e}} F(f_g)\delta_{g,Y}^{-1}) \in \mathrm{rad}_{\mathcal{B}}(F(Y), F(Y))$, which is impossible. Therefore, f_e is non-invertible. As a consequence, $f_e \in \mathrm{rad}_{\mathcal{A}}(Y, Y)$ since $\mathrm{End}_{\mathcal{A}}(Y)$ is local. By the assumption, we know that $\mathrm{Im}\mathcal{A}(Y, u) = \mathrm{rad}_{\mathcal{A}}(Y, Y)$. Thus, there exists a morphism $h_e : Y \to X$ such that $f_e = uh_e$.

Set $h = \sum_{g\in G} F(h_g)\delta_{g,Y}^{-1} : F(Y) \to F(X)$. Then $f = F(u)h \in \mathrm{Im}\mathcal{B}(F(Y), F(u))$. It completes the proof. □

Lemma 4.5. *Let $\mathcal{A}$, $\mathcal{B}$ be two Krull-Schmidt categories with G a group acting freely on $\mathcal{A}$ and $F : \mathcal{A} \to \mathcal{B}$ a normal G-covering. Assume that $u : X \to Y$ is a morphism in $\mathcal{A}$. Then u is right (resp. left) minimal if and only if $F(u)$ is right (resp. left) minimal.*

Proof. For the sufficiency, we assume that $F(u)$ is right minimal with $\mathrm{End}_{\mathcal{B}}(F(Y))$ a local ring. For any $h : X \to X$ with $u = uh$, we have that $F(u) = F(u)F(h)$. Since $F(u)$ is right minimal, we have that $F(h) \in \mathrm{Aut}(F(X))$. By Lemma 2.8 (2), $h \in \mathrm{Aut}(X)$. Hence, u is right minimal.

Assume that u is right minimal. Since $\mathcal{B}$ is Krull-Schmidt, by Proposition 4.2, there exists a decomposition

$$F(u) = [\,a\ b\,] : F(X) = M' \oplus M'' \longrightarrow F(Y)$$

such that a is right minimal and $b = 0$. Since $\mathcal{B}$ is a Krull-Schmidt category and F is dense, we may assume that $M' = F(X_1') \oplus F(X_2') \oplus \cdots \oplus F(X_n')$ and $M'' = F(X_1'') \oplus F(X_2'') \oplus \cdots \oplus F(X_m'')$, where all X_i' and X_j'' are indecomposable in $\mathcal{A}$ such that $F(X_i')$ and $F(X_j'')$ are indecomposable in $\mathcal{B}$ for $1 \le i \le n$ and $1 \le j \le m$. By Definition 2.9 (2), we may assume that

$$X = g_1X_1' \oplus g_2X_2' \oplus \cdots \oplus g_nX_n' \oplus h_1X_1'' \oplus h_2X_2'' \oplus \cdots \oplus h_mX_m'',$$

with all g_i, $h_j \in G$. In this case, we may write

$$u = [\, a' \; b' \,] : (g_1X_1' \oplus g_2X_2' \oplus \cdots \oplus g_nX_n') \oplus (h_1X_1'' \oplus h_2X_2'' \oplus \cdots \oplus h_mX_m'') \longrightarrow Y,$$

where $a' : g_1X_1' \oplus g_2X_2' \oplus \cdots \oplus g_nX_n' \longrightarrow Y$ and $b' : h_1X_1'' \oplus h_2X_2'' \oplus \cdots \oplus h_mX_m'' \longrightarrow Y$. Note that $\delta_{g_i} : Fg_i \to F$ and $\delta_{h_j} : Fh_j \to F$ are functorial isomorphisms, for any $1 \le i \le n$ and $1 \le j \le m$. Set

$$\sigma = \mathrm{diag}\{\delta_{g_1,X_1'}, \delta_{g_2,X_2'}, \cdots, \delta_{g_n,X_n'}\}_{n\times n}$$
$$\tau = \mathrm{diag}\{\delta_{h_1,X_1''}, \delta_{h_2,X_2''}, \cdots, \delta_{h_m,X_m''}\}_{m\times m}.$$

Then $F(u) = [\, F(a') \; F(b') \,] = [\, a \; b \,] \left[\begin{smallmatrix} \sigma & 0 \\ 0 & \tau \end{smallmatrix}\right]$. Hence, $F(a') = a\sigma$ and $F(b') = b\tau = 0$. By Lemma 2.6, F is faithful, and thus $b' = 0$. Note that u is right minimal. Then so does b'. It implies that the zero morphism from $h_1X_1'' \oplus h_2X_2'' \oplus \cdots \oplus h_mX_m''$ to $h_1X_1'' \oplus h_2X_2'' \oplus \cdots \oplus h_mX_m''$ is an automorphism. Thus, $h_1X_1'' \oplus h_2X_2'' \oplus \cdots \oplus h_mX_m''$ is a zero object in $\mathcal{A}$. This yields that all X_j'' are zero objects in $\mathcal{A}$ for $1 \le j \le m$. Therefore, M'' is a zero object in $\mathcal{B}$. Thus, $F(u) = a$ is right minimal. □

Theorem 4.6. *Let $\mathcal{A}$, $\mathcal{B}$ be two Krull-Schmidt categories with G a group acting freely on $\mathcal{A}$ and $F : \mathcal{A} \to \mathcal{B}$ a normal G-covering. Assume that $u : X \to Y$ is a morphism in $\mathcal{A}$. Then u is a sink (resp. source) morphism if and only if $F(u)$ is a sink (resp. source) morphism.*

Proof. It follows from Lemmas 4.3, 4.4 and 4.5. □

Remark 4.7. This result recovers [9, Proposition 3.5], in which the assumption that the G-action on $\mathcal{A}$ is locally bounded, is needed.

Corollary 4.8. *Let k be an algebraically closed field. Let $\mathcal{A}$, $\mathcal{B}$ be two Hom-finite Krull-Schmidt triangulated k-linear categories with G a group acting freely on $\mathcal{A}$ and $F : \mathcal{A} \to \mathcal{B}$ a normal G-covering. If $\mathcal{A}$ has Serre functor, then $\mathcal{B}$ has Serre functor.*

Proof. From [28, Theorem I.2.4] and [20], it suffices to show that for any indecomposable object $Z \in \mathcal{B}$, Z has sink and source morphisms. Since F is a normal G-covering, there is an indecomposable object $Z' \in \mathcal{A}$ such that $F(Z') \cong Z$. Note that $\mathcal{A}$ has Serre functor. Then, there are a source morphism $u : Z' \to Y$ and a sink morphism $v : X \to Z'$ in $\mathcal{A}$. By Theorem 4.6, $F(u)$ is a source morphism and $F(v)$ is a sink morphism. It completes the proof. □

5. Galois *G*-covering of the stable categories

Let $\mathcal{C}$ be an additive category. We recall that $\mathcal{C}$ *has direct sums* provided that any set-indexed family of objects in $\mathcal{C}$ has direct sum. Let $\{X_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$, where I is an index set. If $\mathcal{C}$ has direct sums, then the direct sum of $\{X_i\}_{i\in I}$ exists with the canonical injection $q_j : X_j \to \oplus_{i\in I} X_i$ for each $j \in I$. By definition of the direct sum, there is a unique morphism $p_j : \oplus_{i\in I} X_i \to X_j$, for each $j \in I$, called the *pseudo-projection*, such that

$$p_i q_j = \begin{cases} id_{X_i}, & i = j \\ 0, & i \neq j \end{cases} \tag{5.1}$$

for all i, $j \in I$.

In what follows, unless specifically stated, we assume that k is an algebraically closed field and $\mathcal{A}$ is a locally bounded k-linear category. We say that a left $\mathcal{A}$-module M is *finite dimensional* if $\sum_{x\in\mathcal{A}_0} \dim_k M(x)$ is finite. We denote by $\mathsf{mod}\mathcal{A}$ the full additive subcategory of $\mathsf{Mod}\mathcal{A}$ consisting of all finite dimensional left $\mathcal{A}$-modules. It is well-known that both $\mathsf{Mod}\mathcal{A}$ and $\mathsf{mod}\mathcal{A}$ are abelian categories. Moreover, $\mathsf{mod}\mathcal{A}$ is a Hom-finite Krull-Schmidt category. Since $\mathcal{A}$ is locally bounded, each projective left $\mathcal{A}$-module $P[x]$ is finite dimensional and each left $\mathcal{A}$-module M in $\mathsf{mod}\mathcal{A}$ has a projective cover $P \to M$, where P is a finite dimensional projective left $\mathcal{A}$-module. It means that each module in $\mathsf{mod}\mathcal{A}$ is finitely presented.

Let M be an $\mathcal{A}$-module in $\mathsf{Mod}\mathcal{A}$. We denote by

$$\mathrm{supp} M = \{x \in \mathcal{A}_0 | M(x) \neq 0\},$$

the *support* of M. Let x be an object of $\mathcal{A}$. Let $\mathcal{A}_x$ denote the full subcategory of $\mathcal{A}$ formed by the objects of all $\mathrm{supp} M$, where M is indecomposable and $M(x) \neq 0$. A locally bounded k-linear category $\mathcal{A}$ is called *locally support finite* if for every $x \in \mathcal{A}_0$, $\mathcal{A}_x$ is finite.

The full subcategory of $\mathsf{Mod}\mathcal{A}$ consisting of projective objects is denoted by $\mathsf{Prj}\mathcal{A}$. Note that an $\mathcal{A}$-module P is projective if and only if P is isomorphic to a direct summand of a direct sum of representable functors $P[x]$, where $x \in \mathcal{A}_0$. The full subcategory of $\mathsf{Prj}\mathcal{A}$ consisting of finitely generated projective $\mathcal{A}$-modules is denoted by $\mathsf{prj}\mathcal{A}$.

Let M be a finitely generated module in $\mathsf{Mod}\mathcal{A}$. By [4, Proposition 2.1(c)], for any morphism $f : M \to \oplus_{i\in I} N_i$, there is a finite subset J of I such that there is a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f'} & \oplus_{j\in J} N_j \\ & {}_{f}\searrow & \downarrow{\scriptstyle \eta} \\ & & \oplus_{i\in I} N_i, \end{array}$$

where $\eta = (q_j)_{j\in J}$ with $q_j : N_j \to \oplus_{i\in I} N_i$ the canonical injection, for any $j \in J$. Moreover, there is a natural isomorphism $\mathrm{Hom}_{\mathcal{A}}(M, \oplus_{i\in I} N_i) \cong \oplus_{i\in I} \mathrm{Hom}_{\mathcal{A}}(M, N_i)$ for any finitely generated module M.

Let G be a group. The G-action on $\mathcal{A}$ induces a G-action on $\mathsf{Mod}\mathcal{A}$. Fix $g \in G$. Regarding g as an automorphism of $\mathcal{A}$, for each left $\mathcal{A}$-module M, one can define $g \cdot M = M \circ g^{-1} : \mathcal{A} \to \mathsf{Mod}k$ and for any morphism $f \in \mathrm{Hom}_{\mathcal{A}}(M, N)$, one can define $g \cdot f : g \cdot M \to g \cdot N$ given by $g \cdot f(x) = f(g^{-1}x)$ for any $x \in \mathcal{A}_0$. In particular, $g \cdot P[x] = P[gx]$, for any $x \in \mathcal{A}_0$ and $g \in G$.

By Bongartz and Gabriel's classical construction in [11], each G-invariant Galois G-covering $\pi : \mathcal{A} \to \mathcal{B}$ induces an adjoint triple $(\pi_\bullet, \pi^\bullet, \pi_\diamond)$ between $\mathsf{Mod}\mathcal{A}$ and $\mathsf{Mod}\mathcal{B}$. We will describe $(\pi_\bullet, \pi^\bullet)$ explicitly.

Now, we assume that the G-action is free. Let $\pi : \mathcal{A} \to \mathcal{B}$ be a G-invariant Galois G-covering. Now we recall from [11] the *push-down* functor $\pi_\bullet : \mathsf{Mod}\mathcal{A} \to \mathsf{Mod}\mathcal{B}$. For any $M \in \mathsf{Mod}\mathcal{A}$, the left $\mathcal{B}$-module $\pi_\bullet(M)$ is defined as follows. For any $b \in \mathcal{B}_0$,

$$\pi_\bullet(M)(b) := \oplus_{a \in \pi^{-1}(b)} M(a),$$

where $\pi^{-1}(b) = \{a \in \mathcal{A}_0 | \pi(a) = b\}$. Let $\alpha : x \to y$ be a morphism in $\mathcal{B}$. Since $\pi : \mathcal{A} \to \mathcal{B}$ is a G-invariant Galois G-covering, for any $a \in \pi^{-1}(x)$, there is an isomorphism

$$\oplus_{b \in \pi^{-1}(y)} \mathcal{A}(a, b) \cong \mathcal{B}(x, y)$$

induced by π. For each pair $(a, b) \in \pi^{-1}(x) \times \pi^{-1}(y)$, there is a unique family $\{\alpha_{b,a} : a \to b\}_{b \in \pi^{-1}(y)}$ such that $\sum_{b \in \pi^{-1}(y)} \pi(\alpha_{b,a}) = \alpha$. Then one defines

$$\pi_\bullet(M)(\alpha) := (M(\alpha_{b,a}))_{(b,a) \in \pi^{-1}(y) \times \pi^{-1}(x)} : \oplus_{a \in \pi^{-1}(x)} M(a) \to \oplus_{b \in \pi^{-1}(y)} M(b).$$

For any morphism $f : M \to N$ in $\mathsf{Mod}\mathcal{A}$, one defines $\pi_\bullet(f) : \pi_\bullet(M) \to \pi_\bullet(N)$ as follows:

$$\pi_\bullet(f)(b) := \mathrm{diag}\{f(x) | x \in \pi^{-1}(b)\} : \oplus_{x \in \pi^{-1}(b)} M(x) \to \oplus_{x \in \pi^{-1}(b)} N(x).$$

Following [11,17] and [9, Lemma 6.3], the push-down functor $\pi_\bullet : \mathsf{Mod}\mathcal{A} \to \mathsf{Mod}\mathcal{B}$ is exact and admits a G-stabilizer δ. For any $g \in G$ and $M \in \mathsf{Mod}\mathcal{A}$, the functorial isomorphism $\delta_{g,M} : \pi_\bullet(g \cdot M) \to \pi_\bullet(M)$ is defined as follows: for any $b \in \mathcal{B}_0$, one defines

$$\delta_{g,M}(b) := (\varepsilon_{y,x})_{(y,x) \in \pi^{-1}(b) \times \pi^{-1}(b)} : \oplus_{x \in \pi^{-1}(b)} M(g^{-1}x) \to \oplus_{y \in \pi^{-1}(b)} M(y),$$

where $\varepsilon_{y,x} : M(g^{-1}x) \to M(y)$ is a k-linear map such that

$$\varepsilon_{y,x} = \begin{cases} id_y, & g^{-1}x = y; \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, $\pi_\bullet(P[x]) = P[\pi(x)]$ for any $x \in \mathcal{A}_0$.

We also recall the *pull-up* functor, denoted by

$$\pi^\bullet : \mathsf{Mod}\mathcal{B} \to \mathsf{Mod}\mathcal{A},$$

which is an exact functor, as well as a right adjoint functor of $\pi_\bullet$. For any $N \in \mathsf{Mod}\mathcal{B}$, $\pi^\bullet(N) = N \circ \pi$. For any $f : M \to N$ in $\mathsf{Mod}\mathcal{B}$ and $x \in \mathcal{A}_0$, $\pi^\bullet(f)(x) = f(\pi(x))$. By the definition of $\pi^\bullet$ and the fact that $\pi : \mathcal{A} \to \mathcal{B}$ is a G-invariant Galois G-covering, we have that $\pi^\bullet(P[b]) = \mathcal{B}(b, \pi(-)) \cong \oplus_{x\in\pi^{-1}(b)} P[x]$ for any $b \in \mathcal{B}_0$. Moreover, $\pi^\bullet(\mathsf{Prj}\mathcal{B}) \subseteq \mathsf{Prj}\mathcal{A}$. Since the G-action on $\mathcal{A}$ is free, Remark 6.3 together with Theorem 6.2 of [1] imply that $\pi^\bullet$ is fully faithful.

Next, we recall the construction of adjoint isomorphisms ϕ and ψ of $(\pi_\bullet, \pi^\bullet)$, where for any $M \in \mathsf{Mod}\mathcal{A}$ and $N \in \mathsf{Mod}\mathcal{B}$, both

$$\phi_{M,N} : \mathrm{Hom}_{\mathcal{A}}(M, \pi^\bullet(N)) \to \mathrm{Hom}_{\mathcal{B}}(\pi_\bullet(M), N) \tag{5.2}$$

$$\psi_{M,N} : \mathrm{Hom}_{\mathcal{B}}(\pi_\bullet(M), N) \to \mathrm{Hom}_{\mathcal{A}}(M, \pi^\bullet(N)) \tag{5.3}$$

are isomorphisms and natural in two variables M, N. For any $u : M \to \pi^\bullet(N)$ and $b \in \mathcal{B}_0$, $\phi_{M,N}(u)(b) := (u(x))_{x\in\pi^{-1}(b)} : \oplus_{x\in\pi^{-1}(b)} M(x) \to N(b)$ and $\phi_{M,N}(u) := (\phi_{M,N}(u)(b))_{b\in\mathcal{B}_0}$.

It is easy to check that for any morphisms $u \in \mathrm{Hom}_{\mathcal{A}}(M, \pi^\bullet(N))$ and $v \in \mathrm{Hom}_{\mathcal{B}}(\pi_\bullet(M), N)$,

$$\phi_{M,N}(u) = \lambda_N \circ \pi_\bullet(u),$$
$$\psi_{M,N}(v) = \pi^\bullet(v) \circ \mu_M,$$

where $\lambda_N = \phi_{\pi^\bullet(N),N}(id_{\pi^\bullet(N)}) : \pi_\bullet\pi^\bullet(N) \to N$ and $\mu_M = \psi_{M,\pi_\bullet(M)}(id_{\pi_\bullet(M)}) : M \to \pi^\bullet\pi_\bullet(M)$ are the counit and the unit of $(\pi_\bullet, \pi^\bullet)$, respectively.

Let $\mathcal{C}$ be an additive k-linear category. Recall that $\mathcal{I}$ is said to be an ideal on $\mathcal{C}$ if $\mathcal{I}(x, y)$ is a k-submodule of $\mathcal{C}(x, y)$ and for any $f \in \mathcal{I}(x, y)$, $g \in \mathcal{C}(z, x)$ and $h \in \mathcal{C}(y, s)$, $fg \in \mathcal{I}(z, y)$ and $hf \in \mathcal{I}(x, s)$. Then the *quotient* category of $\mathcal{C}$, denoted by $\mathcal{C}/\mathcal{I}$, has the same objects as $\mathcal{C}$, and for any two objects $x, y \in \mathcal{C}_0$, $\underline{\mathcal{C}}(x, y) := \mathcal{C}(x, y)/\mathcal{I}(x, y)$ is the quotient module of $\mathcal{C}(x, y)$. Let $\mathcal{Q}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}/\mathcal{I}$ be the quotient functor. One has the following universal property: for any k-linear category $\mathcal{C}'$ and k-linear functor $F : \mathcal{C} \to \mathcal{C}'$ which satisfies $F(X) \cong 0$ for $X \in \mathcal{I}$, there exists a unique k-linear functor $\overline{F} : \mathcal{C}/\mathcal{I} \to \mathcal{C}'$ such that $F = \overline{F} \circ \mathcal{Q}_{\mathcal{C}}$.

Example 5.1. Let $\mathcal{C}$ be an additive k-linear category. Assume that $\mathcal{I}$ is a full subcategory of $\mathcal{C}$ which is closed under taking finite direct sums and direct summands (i.e., for any two objects $x, y \in \mathcal{C}_0$, $x \oplus y \in \mathcal{I}$ if and only if x, $y \in \mathcal{I}$) and $\mathcal{I}(x, y)$ is a k-submodule of $\mathcal{C}(x, y)$ consisting of morphisms factoring through some object in $\mathcal{I}$. Then, in this case, $\mathcal{I}$ is an ideal on $\mathcal{C}$, see [16, Lemma 4.3].

Let $\mathcal{A}$ and $\mathcal{B}$ be locally bounded categories with G a group acting freely on $\mathcal{A}$ and $\pi : \mathcal{A} \to \mathcal{B}$ a G-invariant Galois G-covering. Let $\mathcal{C}_{\mathcal{A}}$ and $\mathcal{C}_{\mathcal{B}}$ be full subcategories of $\mathsf{mod}\mathcal{A}$ and $\mathsf{mod}\mathcal{B}$, respectively, which are both closed under taking finite direct sums and direct summands. Let $\mathcal{D}_{\mathcal{A}}$ and $\mathcal{D}_{\mathcal{B}}$ be the additive full subcategories of $\mathsf{Mod}\mathcal{A}$ and $\mathsf{Mod}\mathcal{B}$, respectively, such that $\mathcal{D}_{\mathcal{A}} \bigcap \mathsf{mod}\mathcal{A} = \mathcal{C}_{\mathcal{A}}$ and $\mathcal{D}_{\mathcal{B}} \bigcap \mathsf{mod}\mathcal{B} = \mathcal{C}_{\mathcal{B}}$ and both of them are closed under any direct sums and direct summands. If $\mathsf{prj}\mathcal{A} \subseteq \mathcal{C}_{\mathcal{A}}$ and $\mathsf{prj}\mathcal{B} \subseteq \mathcal{C}_{\mathcal{B}}$, then $\mathsf{Prj}\mathcal{A} \subseteq \mathcal{D}_{\mathcal{A}}$ and $\mathsf{Prj}\mathcal{B} \subseteq \mathcal{D}_{\mathcal{B}}$ since $\mathsf{Prj}\mathcal{A} = \mathrm{Add}(\mathsf{prj}\mathcal{A})$ and $\mathsf{Prj}\mathcal{B} = \mathrm{Add}(\mathsf{prj}\mathcal{B})$. In this case, from Example 5.1, we see that $\mathsf{Prj}\mathcal{A}$ and $\mathsf{prj}\mathcal{A}$ are ideals on $\mathcal{D}_{\mathcal{A}}$ and $\mathcal{C}_{\mathcal{A}}$, respectively. We define the *stable categories* of $\mathcal{D}_{\mathcal{A}}$ and $\mathcal{C}_{\mathcal{A}}$, denoted by $\underline{\mathcal{D}}_{\mathcal{A}}$ and $\underline{\mathcal{C}}_{\mathcal{A}}$ the quotient categories $\mathcal{D}_{\mathcal{A}}/\mathsf{Prj}\mathcal{A}$ and $\mathcal{C}_{\mathcal{A}}/\mathsf{prj}\mathcal{A}$, respectively. It is easy to see that $\underline{\mathcal{C}}_{\mathcal{A}}$ is a full subcategory of $\underline{\mathcal{D}}_{\mathcal{A}}$. By abuse of notation, the quotient functors $\mathcal{C}_{\mathcal{A}} \to \underline{\mathcal{C}}_{\mathcal{A}}$ and $\mathcal{D}_{\mathcal{A}} \to \underline{\mathcal{D}}_{\mathcal{A}}$ are both denoted by $\mathcal{Q}_{\mathcal{A}}$.

Let $\mathcal{K}$ be a k-linear category with a G-action. Recall that $\mathcal{C}$ is said to be a *G-subcategory* of $\mathcal{K}$ if $\mathcal{C}$ is a full subcategory of $\mathcal{K}$ and $g \cdot \mathcal{C}_{\mathcal{A}} \subseteq \mathcal{C}_{\mathcal{A}}$ for any $g \in \mathcal{A}$. For example, $\mathsf{Prj}\mathcal{A}$ and $\mathsf{prj}\mathcal{A}$ are G-subcategories of $\mathsf{Mod}\mathcal{A}$ and $\mathsf{mod}\mathcal{A}$, respectively.

Moreover, if $\mathcal{D}_{\mathcal{A}}$ is a G-subcategory of $\mathsf{Mod}\mathcal{A}$, then so is $\mathcal{C}_{\mathcal{A}}$ in $\mathsf{mod}\mathcal{A}$ since $\mathcal{D}_{\mathcal{A}} \bigcap \mathsf{mod}\mathcal{A} = \mathcal{C}_{\mathcal{A}}$. Then, in this case, the G-actions on $\mathsf{mod}\mathcal{A}$ and $\mathsf{Mod}\mathcal{A}$ restrict to $\mathcal{C}_{\mathcal{A}}$ and $\mathcal{D}_{\mathcal{A}}$.

Notation 1. Let $\mathcal{A}$ and $\mathcal{B}$ be locally bounded categories with a group G acting freely on $\mathcal{A}$ and $\pi : \mathcal{A} \to \mathcal{B}$ a G-invariant Galois G-covering. We fix the following notations:

(1) Denote by $\mathcal{C}_{\mathcal{A}}$ and $\mathcal{C}_{\mathcal{B}}$ the full subcategories of $\mathsf{mod}\mathcal{A}$ and $\mathsf{mod}\mathcal{B}$ containing $\mathsf{prj}\mathcal{A}$ and $\mathsf{prj}\mathcal{B}$, and closed under taking finite direct sums and direct summands, respectively.
(2) Denote by $\mathcal{D}_{\mathcal{A}}$ and $\mathcal{D}_{\mathcal{B}}$ the additive full subcategories of $\mathsf{Mod}\mathcal{A}$ and $\mathsf{Mod}\mathcal{B}$, respectively, such that $\mathcal{D}_{\mathcal{A}} \bigcap \mathsf{mod}\mathcal{A} = \mathcal{C}_{\mathcal{A}}$ and $\mathcal{D}_{\mathcal{B}} \bigcap \mathsf{mod}\mathcal{B} = \mathcal{C}_{\mathcal{B}}$ and both of them are closed under any direct sums and direct summands.

Consequently, we have the following two lemmas. Their proofs are trivial and omitted.

Lemma 5.2. *Keep the notations in* Notation 1. *Assume that $\mathcal{D}_{\mathcal{A}}$ is a G-subcategory of* $\mathsf{Mod}\mathcal{A}$. *Regard $g \in G$ as an automorphism of $\mathcal{D}_{\mathcal{A}}$ and $\mathcal{C}_{\mathcal{A}}$. Then g is also an automorphism of $\underline{\mathcal{D}}_{\mathcal{A}}$ and $\underline{\mathcal{C}}_{\mathcal{A}}$ with the following commutative diagram*

$$\begin{array}{ccc} \mathcal{C}_{\mathcal{A}} & \xrightarrow{\mathcal{Q}_{\mathcal{A}}} & \underline{\mathcal{C}}_{\mathcal{A}} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} \\ \mathcal{C}_{\mathcal{A}} & \xrightarrow{\mathcal{Q}_{\mathcal{A}}} & \underline{\mathcal{C}}_{\mathcal{A}} \end{array} \qquad \begin{array}{ccc} \mathcal{D}_{\mathcal{A}} & \xrightarrow{\mathcal{Q}_{\mathcal{A}}} & \underline{\mathcal{D}}_{\mathcal{A}} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} \\ \mathcal{D}_{\mathcal{A}} & \xrightarrow{\mathcal{Q}_{\mathcal{A}}} & \underline{\mathcal{D}}_{\mathcal{A}}. \end{array}$$

In this case, both $\underline{\mathcal{C}'}_{\mathcal{A}}$ and $\underline{\mathcal{C}}_{\mathcal{A}}$ have G-actions.

Lemma 5.3. *Keep the notations in* Notation 1. *If $\pi_\bullet$ sends $\mathcal{D}_{\mathcal{A}}$ to $\mathcal{D}_{\mathcal{B}}$ and $\pi^\bullet$ sends $\mathcal{D}_{\mathcal{B}}$ to $\mathcal{D}_{\mathcal{A}}$, then $\pi_\bullet$ and $\pi^\bullet$ induce the functors $\overline{\pi_\bullet} : \underline{\mathcal{D}}_{\mathcal{A}} \to \underline{\mathcal{D}}_{\mathcal{B}}$ and $\overline{\pi^\bullet} : \underline{\mathcal{D}}_{\mathcal{B}} \to \underline{\mathcal{D}}_{\mathcal{A}}$ with the following commutative diagrams*

$$\begin{array}{ccc} \mathcal{D}_{\mathcal{A}} & \xrightarrow{Q_{\mathcal{A}}} & \underline{\mathcal{D}}_{\mathcal{A}} \\ \downarrow{\scriptstyle \pi_\bullet} & & \downarrow{\scriptstyle \overline{\pi_\bullet}} \\ \mathcal{D}_{\mathcal{B}} & \xrightarrow{Q_{\mathcal{B}}} & \underline{\mathcal{D}}_{\mathcal{B}} \end{array} \qquad \begin{array}{ccc} \mathcal{D}_{\mathcal{B}} & \xrightarrow{Q_{\mathcal{B}}} & \underline{\mathcal{D}}_{\mathcal{B}} \\ \downarrow{\scriptstyle \pi^\bullet} & & \downarrow{\scriptstyle \overline{\pi^\bullet}} \\ \mathcal{D}_{\mathcal{A}} & \xrightarrow{Q_{\mathcal{A}}} & \underline{\mathcal{D}}_{\mathcal{A}}. \end{array}$$

Moreover, $\overline{\pi_\bullet}$ can be restricted to $\underline{\mathcal{C}}$, which, by abuse of notation, is denoted by $\overline{\pi_\bullet}$ again.

Proof. It suffices to show that $\overline{\pi_\bullet}$ can be restricted to $\underline{\mathcal{C}}$. Indeed, since $\mathcal{D}_{\mathcal{A}} \bigcap \mathsf{mod}\mathcal{A} = \mathcal{C}_{\mathcal{A}}$, $\mathcal{D}_{\mathcal{B}} \bigcap \mathsf{mod}\mathcal{B} = \mathcal{C}_{\mathcal{B}}$ and $\pi_\bullet$ preserves finitely generated modules, we have that $\pi_\bullet$ can be restricted to $\mathcal{C}$. Thus, $\overline{\pi_\bullet}$ can be restricted to $\underline{\mathcal{C}}$. □

Proposition 5.4. *Keep the notations in* Notation 1. *Assume that $\mathcal{D}_{\mathcal{A}}$ is a G-subcategory of* $\mathsf{Mod}\mathcal{A}$. *If $\pi_\bullet$ sends $\mathcal{D}_{\mathcal{A}}$ to $\mathcal{D}_{\mathcal{B}}$ and $\pi^\bullet$ sends $\mathcal{D}_{\mathcal{B}}$ to $\mathcal{D}_{\mathcal{A}}$, then the following hold.*

(1) $(\overline{\pi_\bullet}, \overline{\pi^\bullet})$ *is an adjoint pair between $\underline{\mathcal{D}}_{\mathcal{A}}$ and $\underline{\mathcal{D}}_{\mathcal{B}}$.*
(2) *For any X, $Y \in \underline{\mathcal{C}}_{\mathcal{A}}$, there is a natural isomorphism*

$$\oplus_{g\in G}\underline{\mathcal{C}}_{\mathcal{A}}(X, gY) \cong \underline{\mathcal{D}}_{\mathcal{A}}(X, \oplus_{g\in G} gY).$$

Proof. (1) Fix $M \in \mathcal{D}_{\mathcal{A}}$ and $N \in \mathcal{D}_{\mathcal{B}}$. Then we have the adjoint isomorphisms $\phi_{M,N}$ and $\psi_{M,N}$ defined as (5.2) and (5.3). Since $\mathcal{D}_{\mathcal{A}}$ and $\mathcal{D}_{\mathcal{B}}$ are full subcategories of $\mathsf{Mod}\mathcal{A}$ and $\mathsf{Mod}\mathcal{B}$, we have that $\phi_{M,N} : \mathcal{D}_{\mathcal{A}}(M, \pi^\bullet(N)) \to \mathcal{D}_{\mathcal{B}}(\pi_\bullet(M), N)$ and $\psi_{M,N} : \mathcal{D}_{\mathcal{B}}(\pi_\bullet(M), N) \to \mathcal{D}_{\mathcal{A}}(M, \pi^\bullet(N))$ are isomorphisms. For any $f \in \mathsf{Prj}\mathcal{A}(M, \pi^\bullet(N))$, there is a projective $\mathcal{A}$-module P such that f factors through P. Since $\phi_{M,N}(f) = \lambda_N \circ \pi_\bullet(f)$, where λ_N is a counit. Note that $\pi_\bullet$ preserves projective objects. Thus, $\phi_{M,N}(f) = \lambda_N \circ \pi_\bullet(f) \in \mathsf{Prj}\mathcal{B}(\pi_\bullet(M), N)$. Similarly, since $\pi^\bullet$ preserves projective objects, $\psi_{M,N}(g) = \pi^\bullet(g) \circ \mu_M \in \mathsf{Prj}\mathcal{A}(M, \pi^\bullet(N))$ for any $g \in \mathsf{Prj}\mathcal{B}(\pi_\bullet(M), N)$, where $\mu_M = \psi_{M,\pi_\bullet(M)}(id_{\pi_\bullet(M)})$ is a unit. Then the restriction of $\phi_{M,N}$ on $\mathsf{Prj}\mathcal{A}(M, \pi^\bullet(N))$ is an isomorphism. Therefore, there is a natural isomorphism

$$\underline{\mathcal{D}}_{\mathcal{A}}(M, \pi^\bullet(N)) \cong \underline{\mathcal{D}}_{\mathcal{B}}(\pi_\bullet(M), N),$$

where the naturality comes from that of $\phi_{M,N}$.

(2) Given X, $Y \in \underline{\mathcal{C}}_{\mathcal{A}}$, we see that X, Y are finitely generated $\mathcal{A}$-modules. For any $(u_g)_{g\in G} \in \oplus_{g\in G}\mathrm{Hom}_{\mathcal{A}}(X, gY)$, there is a finite subset G_0 of G such that $u_h = 0$ for any $h \in G\backslash G_0$. Then we have the morphism $f = \eta_{G_0} \circ f' \in \mathrm{Hom}_{\mathcal{A}}(X, \oplus_{g\in G} gY)$, where $f' = (u_h)_{h\in G} : X \to \oplus_{h\in G_0} hY$ is a column-matrix, $\eta_{G_0} = (q_h)_{h\in G_0}$ is a matrix with $q_h : N_h \to \oplus_{g\in G} gY$ the canonical injection, for $h \in G_0$. From [4, Proposition 2.1(c)], we have the natural isomorphism

$$\theta_{X,Y} : \oplus_{g\in G}\mathrm{Hom}_{\mathcal{A}}(X, gY) \to \mathrm{Hom}_{\mathcal{A}}(X, \oplus_{g\in G} gY),$$

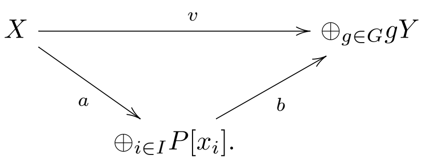
which sends $(u_g)_{g\in G}$ to $f = \eta_{G_0} \circ f'$ by the discussion above. Since $\mathcal{D}_{\mathcal{A}}$ and $\mathcal{C}_{\mathcal{A}}$ are full G-subcategories of $\mathsf{Mod}\mathcal{A}$ and $\mathcal{D}_{\mathcal{A}}$ is closed under any direct sums, we have the natural isomorphism

$$\theta_{X,Y} : \oplus_{g\in G}\mathcal{C}_{\mathcal{A}}(X, gY) \to \mathcal{D}_{\mathcal{A}}(X, \oplus_{g\in G} gY).$$

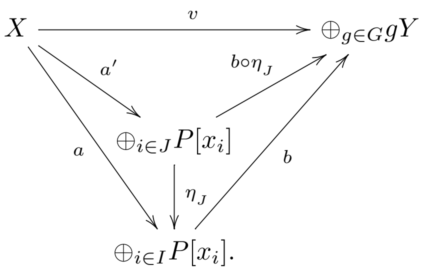
Next, we show that the restriction of $\theta_{X,Y}$ on $\oplus_{g\in G}\mathsf{prj}\mathcal{A}(X, gY)$, denoted by $\theta_{X,Y}|$, is an isomorphism. For any $(u_g)_{g\in G} \in \oplus_{g\in G}\mathsf{prj}\mathcal{A}(X, gY)$ with each $u_g \in \mathsf{prj}\mathcal{A}(X, gY)$, there is a finite subset G_0 of G such that $u_h = 0$ for any $h \in G\backslash G_0$. For each $h \in G_0$, since $u_h \in \mathsf{prj}\mathcal{A}(X, hY)$, there are two morphisms $s_h : X \to P_h$ and $t_h : P_h \to hY$ such that $u_h = t_h \circ s_h$, where $P_h \in \mathsf{prj}\mathcal{A}$. Then $f' = (u_h)_{h\in G_0} = \mathrm{diag}\{t_h|h \in G_0\} \circ (s_h)_{h\in G_0}$ and so $\theta_{X,Y}((u_g)_{g\in G}) = \eta_{G_0} \circ f' = \eta_{G_0} \circ \mathrm{diag}\{t_h|h \in G_0\} \circ (s_h)_{h\in G_0}$, where $(s_h)_{h\in G_0}$: $X \to \oplus_{h\in G_0} P_h$ is a column-matrix, and $\mathrm{diag}\{t_h|h \in G_0\} : \oplus_{h\in G_0} P_h \to \oplus_{h\in G_0} hY$ is a diagonal matrix. It implies that $\theta_{X,Y}((u_g)_{g\in G}) \in \mathsf{Prj}\mathcal{A}(X, \oplus_{g\in G} gY)$. Since $\theta_{X,Y}$ is injective, the restriction map $\theta_{X,Y}| : \oplus_{g\in G}\mathsf{prj}\mathcal{A}(X, gY) \to \mathsf{Prj}\mathcal{A}(X, \oplus_{g\in G} gY)$ is injective. For any $v \in \mathsf{Prj}\mathcal{A}(X, \oplus_{g\in G} gY)$, since X is finitely generated, there is a finite subset G_0 of G such that $v = \eta_{G_0} v'$, where $\eta_{G_0} = (q_h)_{h\in G_0}$ is a matrix with $q_h : N_h \to \oplus_{g\in G} gY$ the canonical injection, for $h \in G_0$, and $v' : X \to \oplus_{h\in G_0} hY$. We may write $v' = (v'_h)_{h\in G_0}$, where $v'_h : X \to hY$ for $h \in G_0$. Then $v = (q_h v'_h)_{h\in G_0}$ and $\theta_{X,Y}((v'_h)_{h\in G_0}) = v$.

Next, we shall prove that $v'_h \in \mathsf{prj}\mathcal{A}(X, hY)$ for each $h \in G_0$.

Since $v \in \mathsf{Prj}\mathcal{A}(X, \oplus_{g\in G} gY)$, we assume that there is a projective module $\oplus_{i\in I} P[x_i]$ such that v factors through $\oplus_{i\in I} P[x_i]$. Consider the commutative diagram



For the morphism $a : X \to \oplus_{i\in I} P[x_i]$, since X is finitely generated, there is a finite subset J of I such that there is a commutative diagram



Set $p_k : \oplus_{g\in G} gY \to kY$, for $k \in G_0$. Then for each $k \in G_0$, $p_k v = (p_k q_h v'_h)_{h\in G_0} = v'_k$ since $p_k q_h = 1$ whenever $h = k$. Then there is a commutative diagram

$$\begin{array}{ccccc} & & \overset{v'_k}{\curvearrowright} & & \\ X & \xrightarrow{v} & \oplus_{g\in G} gY & \xrightarrow{p_k} & kY. \\ & {\scriptstyle a'}\searrow & \nearrow{\scriptstyle b\circ\eta_J} & \nearrow{\scriptstyle p_k\circ b\circ\eta_J} & \\ & & \oplus_{i\in J} P[x_i] & & \end{array}$$

Thus $v'_k \in \mathsf{prj}\mathcal{A}(X, kY)$ since $\oplus_{i\in J} P[x_i] \in \mathsf{prj}\mathcal{A}$ for each $k \in G_0$. It implies that $\theta_{X,Y}|$ is surjective, and hence $\theta_{X,Y}|$ is an isomorphism. Then we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \oplus_{g\in G}\mathsf{prj}\mathcal{A}(X, gY) & \longrightarrow & \oplus_{g\in G}\mathcal{C}_{\mathcal{A}}(X, gY) & \longrightarrow & \oplus_{g\in G}\underline{\mathcal{C}}_{\mathcal{A}}(X, gY) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \theta_{X,Y}|} & & \downarrow{\scriptstyle \theta_{X,Y}} & & \downarrow{\scriptstyle \overline{\theta}_{X,Y}} & & \\ 0 & \longrightarrow & \mathsf{Prj}\mathcal{A}(X, \oplus_{g\in G} gY) & \longrightarrow & \mathcal{D}_{\mathcal{A}}(X, \oplus_{g\in G} gY) & \longrightarrow & \underline{\mathcal{C}}_{\mathcal{A}}(X, \oplus_{g\in G} gY) & \longrightarrow & 0. \end{array}$$

By Five Lemma, we know that $\overline{\theta}_{X,Y}$ is an isomorphism. The naturality of $\overline{\theta}_{X,Y}$ follows from $\theta_{X,Y}$. □

Theorem 5.5. *Keep the notations in* Notation 1. *Assume that $\mathcal{D}_{\mathcal{A}}$ is a G-subcategory of* $\mathsf{Mod}\mathcal{A}$. *If $\pi_\bullet$ sends $\mathcal{D}_{\mathcal{A}}$ to $\mathcal{D}_{\mathcal{B}}$ and $\pi^\bullet$ sends $\mathcal{D}_{\mathcal{B}}$ to $\mathcal{D}_{\mathcal{A}}$, then the following hold.*

(1) *$\overline{\pi_\bullet} : \underline{\mathcal{C}}_{\mathcal{A}} \to \underline{\mathcal{C}}_{\mathcal{B}}$ is a G-precovering.*
(2) *If $\mathcal{A}$ is locally support-finite and G is torsion-free, then $\overline{\pi_\bullet} : \underline{\mathcal{C}}_{\mathcal{A}} \to \underline{\mathcal{C}}_{\mathcal{B}}$ is a Galois G-covering.*

Proof. (1) By [9, Lemma 6.3], there is a G-stabilizer $\delta = \{\delta_g : \pi_\bullet \circ g \to \pi_\bullet \mid g \in G\}$ for $\pi_\bullet$. Since $\mathcal{C}_{\mathcal{A}}$ is a G-subcategory of $\mathsf{Mod}\mathcal{A}$ and $\pi_\bullet$ can be restricted to $\mathcal{C}$, the restriction of δ_g on $\mathcal{C}_{\mathcal{A}}$ is also a functorial isomorphism. For any $N \in \mathcal{C}_{\mathcal{A}}$ and $g \in G$, we define $\underline{\delta}_{g,N} = \mathcal{Q}_{\mathcal{B}}(\delta_{g,N})$. By Lemmas 5.2 and 5.3, $\mathcal{Q}_{\mathcal{B}}(\delta_{g,N}) : \overline{\pi_\bullet}(g \cdot N) \to \overline{\pi_\bullet}(N)$ is an isomorphism in $\underline{\mathcal{C}}_B$. Then we have a functorial isomorphism $\underline{\delta} = \{\underline{\delta}_g\}_{g\in G}$. Moreover, it is easy to check that $\underline{\delta}_{h,N}\underline{\delta}_{g,h\cdot N} = \underline{\delta}_{gh,N}$ for any $g, h \in G$ and $N \in \mathcal{C}_{\mathcal{A}}$. Thus $\underline{\delta} = \{\underline{\delta}_g\}_{g\in G}$ is a G-stabilizer for $\overline{\pi_\bullet}$.

It is well-known that there is a functorial isomorphism $\gamma : \oplus_{g\in G} g \to \pi^\bullet\pi_\bullet$. Since $\mathcal{D}_{\mathcal{A}}$ is a G-subcategory of $\mathsf{Mod}\mathcal{A}$ and $\mathcal{D}_{\mathcal{A}}$ is closed under any direct sums, there is an isomorphism $\gamma_Y : \oplus_{g\in G} gY \to \pi^\bullet\pi_\bullet(Y)$ in $\mathcal{D}_{\mathcal{A}}$. Then it induces an isomorphism $\underline{\gamma}_Y : \oplus_{g\in G} gY \to \overline{\pi^\bullet}\,\overline{\pi_\bullet}(Y)$ in $\underline{\mathcal{D}}_{\mathcal{A}}$ for any $Y \in \mathcal{D}_{\mathcal{A}}$.

For any X and $Y \in \mathcal{C}_{\mathcal{A}}$, from Proposition 5.4 and the fact that $\underline{\mathcal{C}}_{\mathcal{B}}$ is a full subcategory of $\underline{\mathcal{C}}_{\mathcal{B}}$, we have the following commutative diagram

$$\begin{array}{ccc} \oplus_{g\in G}\underline{\mathcal{C}}_{\mathcal{A}}(X,gY) & \xrightarrow{\cong} & \underline{\mathcal{D}}_{\mathcal{A}}(X,\oplus_{g\in G}gY) \\ \downarrow{\scriptstyle \overline{\pi_\bullet}_{X,Y}} & & \cong \downarrow{\scriptstyle \underline{\mathcal{D}}_{\mathcal{A}}(X,\underline{\gamma}_Y)} \\ \underline{\mathcal{C}}_{\mathcal{B}}(\overline{\pi_\bullet}(X),\overline{\pi_\bullet}(Y)) & \xrightarrow{\cong} & \underline{\mathcal{D}}_{\mathcal{A}}(X,\overline{\pi^\bullet}\overline{\pi_\bullet}(Y)). \end{array}$$

Thus $\overline{\pi_\bullet}_{X,Y}$ is an isomorphism. It implies that $\overline{\pi_\bullet}$ is a G-precovering.

(2) Assume that $\mathcal{A}$ is locally support-finite. Then $\pi_\bullet : \mathsf{mod}\mathcal{A} \to \mathsf{mod}\mathcal{B}$ is a G-covering. Let X be a finitely generated $\mathcal{B}$-module in $\mathcal{C}_{\mathcal{B}}$. From [15], there exits a finitely generated $\mathcal{A}$-module X' such that $\pi_\bullet(X') \cong X$ which is a certain direct summand of $\pi^\bullet(X)$. By the assumption, we have that $\pi^\bullet(X) \in \mathcal{D}_{\mathcal{A}}$. Since $\mathcal{D}_{\mathcal{A}}$ is closed under direct summands, we know that $X' \in \mathcal{D}_{\mathcal{A}} \bigcap \mathsf{mod}\mathcal{A}$. Note that $\mathcal{D}_{\mathcal{A}} \bigcap \mathsf{mod}\mathcal{A} = \mathcal{C}_{\mathcal{A}}$. Thus $X' \in \mathcal{C}_{\mathcal{A}}$. Therefore, $\overline{\pi_\bullet}$ is dense and hence, is a G-covering. Since $\mathcal{C}$ is Hom-finite Krull-Schmidt, so is $\underline{\mathcal{C}}$. By [9, Lemma 2.9], $\overline{\pi_\bullet}$ is a Galois G-covering. □

Corollary 5.6 *([19, Proposition 2.6]). Let $\mathcal{A}$ and $\mathcal{B}$ be locally bounded categories with G a group acting freely on $\mathcal{A}$ and $\pi : \mathcal{A} \to \mathcal{B}$ a G-invariant Galois G-covering. Then $\overline{\pi_\bullet} : \underline{\mathsf{mod}}\mathcal{A} \to \underline{\mathsf{mod}}\mathcal{B}$ is a G-precovering. If $\mathcal{A}$ is locally support-finite and G is torsion-free, then $\overline{\pi_\bullet} : \underline{\mathsf{mod}}\mathcal{A} \to \underline{\mathsf{mod}}\mathcal{B}$ is a Galois G-covering.*

Proof. It just takes the notations $\mathcal{C} = \mathsf{mod}$ and $\mathcal{D} = \mathsf{Mod}$. Then it follows from Theorem 5.5. □

Corollary 5.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be locally bounded Frobenius categories with G a torsion-free group acting freely on $\mathcal{A}$ and $\pi : \mathcal{A} \to \mathcal{B}$ a G-invariant Galois G-covering. If $\mathcal{A}$ is locally support-finite, then the triangle functor $\overline{\pi_\bullet} : \underline{\mathsf{mod}}\mathcal{A} \to \underline{\mathsf{mod}}\mathcal{B}$ between two triangulated categories is a Galois G-covering.*

Proof. It follows from [20, Theorem 2.6] and Corollary 5.6. □

Let A be a basic and connected finite-dimensional k-algebra. Then A is a locally bounded k-linear category. The trivial extension $T(A) = A \ltimes DA$ of A by DA is defined to be the k-algebra whose additive structure is that of $A \oplus DA$ and whose multiplicative structure is given by

$$(a, \varphi)(b, \psi) = (ab, a \cdot \psi + \varphi \cdot b),$$

for any $a, b \in A$ and $\varphi, \psi \in DA$. Then $T(A)$ is a self-injective and also is a locally bounded Frobenius category.

The repetitive algebra $\widehat{A}$ of $T(A) = A \ltimes DA$ defined by Hughes and Waschbüsch [21] is the doubly infinite matrix algebra, without identity, which can be represented as

$$\widehat{A} = \begin{bmatrix} \ddots & \ddots & & & & \\ & A_{n-1} & E_{n-1} & & & \\ & & A_n & E_n & & \\ & & & A_{n+1} & E_{n+1} & \\ & & & & \ddots & \ddots \end{bmatrix},$$

in which matrices are assumed to have only finitely many entries different from zero, $A_n = A$ and $E_n = {}_A DA_A$ for all integers n, all the remaining entries are zero, and the multiplication is induced from the canonical maps $A \otimes_A DA \to DA$, $DA \otimes_A A \to DA$, and zero maps $DA \otimes_A DA \to 0$. From [21], $\widehat{A}$ is a Frobenius algebra and always infinite-dimensional. Clearly $\widehat{A}$ is a locally bounded k-algebra. The identity maps $A_n \to A_{n+1}$, $E_n \to E_{n+1}$ induce an automorphism σ of A. The orbit category $\widehat{A}/\langle\sigma\rangle$ inherits from $\widehat{A}$ a k-algebra structure, which is easily checked to be isomorphic to $T(A)$. The projection functor $F : \widehat{A} \to \widehat{A}/\langle\sigma\rangle \cong T(A)$ is a Galois G-covering.

It is shown in [33] that the repetitive algebra $\widetilde{A}$ is locally support-finite if and only if the strong global dimension of $T(A)$ is finite, that is, the indecomposable complexes in the derived category have bounded length.

Corollary 5.8. *Let A be a basic and connected finite-dimensional k-algebra and $T(A) = A \ltimes DA$ the trivial extension of A. If the strong global dimension of $T(A)$ is finite, then the triangle functor $\overline{\pi_\bullet} : \underline{\mathsf{mod}}\widehat{A} \to \underline{\mathsf{mod}}T(A)$ between two triangulated categories is a Galois G-covering.*

Next, we apply our results into Gorenstein homological theory. First, we recall some notions.

Let $\mathcal{E}$ be an abelian category having enough projective objects. We denoted by $\mathsf{Prj}\mathcal{E}$ the full subcategory of $\mathcal{E}$ consisting of projective objects. A complex of projective objects $P^\bullet : \cdots \to P^{i-1} \to P^i \to P^{i+1} \to \cdots$ is said to be a *complete projective complex* provided that the complexes $\mathrm{Hom}_{\mathcal{E}}(P^\bullet, \mathsf{Prj}\mathcal{E})$ and $\mathrm{Hom}_{\mathcal{E}}(\mathsf{Prj}\mathcal{E}, P^\bullet)$ are acyclic. An object X in $\mathcal{A}$ is called *Gorenstein projective* if X is a syzygy of a complete projective complex.

Now, we assume that $\mathcal{A}$ is a locally bounded k-linear category. We denote by $\mathcal{GP}(\mathcal{A})$ the full subcategory of $\mathsf{Mod}\mathcal{A}$ consisting of Gorenstein projective objects in $\mathsf{Mod}\mathcal{A}$. We denote by $\mathcal{G}p(\mathcal{A})$ the full subcategory of $\mathsf{mod}\mathcal{A}$ consisting of finitely generated Gorenstein projective $\mathcal{A}$-modules. It is well-known that $\mathcal{G}p(\mathcal{A})$ is a Frobenius category. By [2, Proposition 4.4], we have that $\mathcal{GP}(\mathcal{A}) \bigcap \mathsf{mod}\mathcal{A} = \mathcal{G}p(\mathcal{A})$.

Corollary 5.9 *([2, Theorem 4.5]). Let $\mathcal{A}$ and $\mathcal{B}$ be locally bounded categories with G a group acting freely on $\mathcal{A}$ and $\pi : \mathcal{A} \to \mathcal{B}$ a G-invariant Galois G-covering. Then the* *triangle functor* $\overline{\pi_\bullet} : \underline{\mathcal{G}p(\mathcal{A})} \to \underline{\mathcal{G}p(\mathcal{B})}$ *is a G-precovering. If* $\mathcal{A}$ *is locally support-finite and G is torsion-free, then* $\overline{\pi_\bullet} : \underline{\mathcal{G}p(\mathcal{A})} \to \underline{\mathcal{G}p(\mathcal{B})}$ *is a Galois G-covering.*

Proof. Set $\mathcal{C} = \mathcal{G}p$ and $\mathcal{D} = \mathcal{GP}$. By [2, Lemmas 3.7 and 4.2], $\pi_\bullet$ sends $\mathcal{D}_\mathcal{A}$ to $\mathcal{D}_\mathcal{B}$ and $\pi^\bullet$ sends $\mathcal{D}_\mathcal{B}$ to $\mathcal{D}_\mathcal{A}$. Then they satisfy the conditions of Theorem 5.5. It is easy to see that $\overline{\pi_\bullet}$ is a triangle functor since $\mathcal{C} = \mathcal{G}p$ is a Frobenius category and $\pi_\bullet$ is exact. □

Corollary 5.10. *Let* $\mathcal{A}$ *be a locally support-finite with G a torsion-free group acting freely on* $\mathcal{A}$*. Let* $\mathcal{B}$ *be a locally bounded category and* $\pi : \mathcal{A} \to \mathcal{B}$ *a G-invariant Galois G-covering. Then the triangulated category* $\underline{\mathcal{G}p(\mathcal{A})}$ *has Serre functor if and only if* $\underline{\mathcal{G}p(\mathcal{B})}$ *has Serre functor.*

Proof. By Corollary 5.9, we have a Galois G-covering $\overline{\pi_\bullet} : \underline{\mathcal{G}p(\mathcal{A})} \to \underline{\mathcal{G}p(\mathcal{B})}$. The statement follows from [9, Theorem 3.7]. □

Data availability

No data was used for the research described in the article.

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